DIVISION OF POLYNOMIALS AND THE REMAINDER THEOREM

If $P(x) = 3x^2 + 7x + 4$, then $P(1)$ is the value of $P(x)$ when $x = 1$, which is found by substituting $x = 1$ into the polynomial. For example:

$P(1) = 3 + 7 + 4 = 14$

$P(-1) = 3 \times (-1)^2 + 7 \times (-1) + 4 = 0$

$P(a) = 3a^2 + 7a + 4$

You already know how to perform long division using integers (e.g. $532 \div 19 = 28$). You can also perform long division of a polynomial by a linear expression or by another polynomial. For example:

$\dfrac{x^2 + 7x + 12}{x + 3}, x \neq -3$

$(x^2 + 7x + 12) \div (x + 3), x \neq -3$

The two ways of writing the division above are equivalent.

Example 3

Find $\dfrac{x^2 + 7x + 12}{x + 3}, x \neq -3$.

Solution

The condition $x \neq -3$ exists because it cannot be divided by zero.

Divide the leading term of the dividend (x^2) by the leading term of the divisor (x) and write the answer (x) above the x^2.

Multiply the divisor $(x + 3)$ by the result of $x^2 \div x$ (i.e. x) to obtain $x^2 + 3x$.

Subtract $(x^2 + 3x)$ from $(x^2 + 7x)$ to get $4x$, then bring down the $+12$ from above.

Divide this new leading term ($4x$) by x and write the answer ($+4$) above $7x$.

Multiply $(x + 3)$ by 4 to obtain $4x + 12$.

Subtract the last two lines to get a remainder of 0, completing the division.

$$
\begin{array}{r}
x + 4 \\
x+3\overline{)x^2+7x+12} \\
\underline{x^2+3x} \\
4x+12 \\
\underline{4x+12} \\
0
\end{array}
$$

Hence you have $\dfrac{x^2 + 7x + 12}{x + 3} = x + 4$, with no remainder.

You could also write $x^2 + 7x + 4 = (x + 3)(x + 4) + 0$:

Dividend = Divisor × Quotient + Remainder

Example 4

If $P(x) = 3x^3 - 7x^2 + 2x + 4$, divide $P(x)$ by $x - 3$.

Solution

Divide $3x^3$ by x and write the answer $(3x^2)$ above the $3x^3$.

Multiply $(x - 3)$ by $3x^2$ to obtain $3x^3 - 9x^2$ and write it under $3x^3 - 7x^2$.

Subtract $(3x^3 - 9x^2)$ from $(3x^3 - 7x^2)$ to get $2x^2$, then bring down the $+2x$ from above.

Divide $(2x^2 + 2x)$ by x and write the answer $(+2x)$ above $-7x^2$.

Multiply $(x - 3)$ by $2x$ to obtain $2x^2 - 6x$ and write it under $2x^2 + 2x$.

Subtract $(2x^2 - 6x)$ from $(2x^2 + 2x)$ to get $8x$, then bring down the $+4$ from above.

Divide $8x$ by x and write the answer $(+8)$ above $+2x$.

Multiply $(x - 3)$ by 8 to obtain $8x - 24$ and write it under $8x + 4$.

Subtract $(8x - 24)$ from $(8x + 4)$ to get 28.

$$\begin{array}{r|l} & 3x^2 + 2x + 8 \\ \hline x-3 & 3x^3 - 7x^2 + 2x + 4 \\ & \underline{3x^3 - 9x^2} \\ & \quad 2x^2 + 2x \\ & \quad \underline{2x^2 - 6x} \\ & \qquad 8x + 4 \\ & \qquad \underline{8x - 24} \\ & \qquad\quad 28 \end{array}$$

The number 28 is a constant with a degree less than the divisor, so it is not divisible by $(x - 3)$. Thus the remainder is 28.

Hence: $\dfrac{3x^3 - 7x^2 + 2x + 4}{x - 3} = 3x^2 + 2x + 8 + \dfrac{28}{x - 3}$

or $3x^3 - 7x^2 + 2x + 4 = (x - 3)(3x^2 + 2x + 8) + 28$

i.e. $P(x) = (x - 3)Q(x) + R$

where $Q(x)$ is the quotient and R is the remainder (i.e. in this case, 28). *Note:*

The degree of the remainder < the degree of the divisor.
The degree of the quotient < the degree of the dividend.

When the divisor is a linear function (first degree), the remainder will be a constant.

Note also that $P(x) = (x - 3)(3x^2 + 2x + 8) + 28$, so:

$$P(3) = (3 - 3) \times Q(3) + 28 = 0 + 28 = 28$$

This suggests that the remainder of division by $(x - 3)$ could be found just by finding the value of $P(3)$, without doing the long division.

Example 5

Divide $4x^3 - 19x + 9$ by $2x - 3$.

Solution

When writing out the dividend, include all missing terms by writing them with zero coefficients.

$$\begin{array}{r|l} & 2x^2 + 3x - 5 \\ \hline 2x-3 & 4x^3 + 0x^2 - 19x + 9 \\ & \underline{4x^3 - 6x^2} \\ & \quad 6x^2 - 19x \\ & \quad \underline{6x^2 - 9x} \\ & \qquad -10x + 9 \\ & \qquad \underline{-10x + 15} \\ & \qquad\qquad -6 \end{array}$$

The quotient is $2x^2 + 3x - 5$ and the remainder is -6.

$$\frac{4x^3 - 19x + 9}{2x - 3} = 2x^2 + 3x - 5 - \frac{6}{2x - 3}$$

or $(4x^3 - 19x + 9) = (2x - 3)(2x^2 + 3x - 5) - 6$

$$P(x) = (2x - 3) \times Q(x) - 6$$

If $2x - 3 = 0$ then $x = \frac{3}{2}$.

$$P\left(\frac{3}{2}\right) = 0 \times Q\left(\frac{3}{2}\right) - 6 = -6$$

This again suggests a simpler method for finding the remainder.

Example 6

Divide $x^4 + x^3 - 7x^2 - x + 6$ by $x^2 - 1$.

Solution

$$
\begin{array}{r}
x^2 + x - 6 \\
x^2 - 1 \overline{\smash{)} x^4 + x^3 - 7x^2 - x + 6} \\
\underline{x^4 - x^2} \\
x^3 - 6x^2 - x \\
\underline{x^3 - x} \\
-6x^2 + 6 \\
\underline{-6x^2 + 6} \\
0
\end{array}
$$

The remainder is zero, so this means that $(x^2 - 1)$ is a factor of $x^4 + x^3 - 7x^2 - x + 6$. This then means that $(x + 1)$ and $(x - 1)$ are also factors of $x^4 + x^3 - 7x^2 - x + 6$.

The remainder theorem

If a polynomial $P(x)$ is divided by $(x - a)$ until the remainder R does not contain x, then $R = P(a)$.

For any polynomial, $P(x) = (x - a)Q(x) + R$, where $Q(x)$ is a polynomial.

Thus $P(a) = (a - a)Q(a) + R$

i.e. $P(a) = 0 \times Q(a) + R$

$\therefore \quad R = P(a)$

Important considerations:

1. If $P(x)$ is divided by $x + a$, as $x + a = x - (-a)$, then $R = P(-a)$.
2. If $P(x)$ is divided by $ax - b$, as $ax - b = a\left(x - \frac{b}{a}\right)$, then $R = P\left(\frac{b}{a}\right)$.

Example 7

Find the remainder when $P(x) = 2x^3 - 6x^2 + 4x + 3$ is divided by the following.

(a) $x - 2$ **(b)** $x + 3$ **(c)** $2x - 1$ **(d)** $3x + 2$

Solution

(a) $R = P(2) = 2 \times 2^3 - 6 \times 2^2 + 4 \times 2 + 3 = 16 - 24 + 8 + 3 = 3$

(b) $R = P(-3) = 2 \times (-3)^3 - 6 \times (-3)^2 + 4 \times (-3) + 3 = -54 - 54 - 12 + 3 = -117$

(c) $R = P\left(\frac{1}{2}\right) = 2 \times \left(\frac{1}{2}\right)^3 - 6 \times \left(\frac{1}{2}\right)^2 + 4 \times \frac{1}{2} + 3 = \frac{1}{4} - \frac{3}{2} + 2 + 3 = 3\frac{3}{4}$

(d) $R = P\left(-\frac{2}{3}\right) = 2 \times \left(-\frac{2}{3}\right)^3 - 6 \times \left(-\frac{2}{3}\right)^2 + 4 \times \left(-\frac{2}{3}\right) + 3 = -\frac{16}{27} - \frac{8}{3} - \frac{8}{3} + 3 = -\frac{79}{27}$