THE LANGUAGE AND LOGIC OF PROOF

Theorems and Conjectures

Mathematical knowledge is based on **theorems**, i.e. significant mathematical statements that have been proved to be true (think about Pythagoras' theorem for example).

Scientific theories are analogous to **mathematical conjectures**, i.e. a statement that mathematicians have reason to believe may be true, but has not been proved definitely. One of the most famous mathematical conjecture is the Goldbach conjecture, named after the eighteenth-century mathematician Christian Goldbach:

Every even integer greater than 2 can be expressed as the sum of two prime numbers.

The Goldbach conjecture seems likely to be true; in fact, it has been shown to be trues for every integer up to 4×10^{18}. However, it is still, at present, unproven. There may, in fact, be a large even integer that cannot be expressed as the sum of two prime numbers.

Mathematical proofs enables mathematics to be a robust system of knowledge that cannot be falsified, and any proven result can be used to help establish further results, adding to this system of knowledge.

Unambiguous language and valid logic to prove mathematical statements

A **mathematical statement** is defined to be an assertion:

1) that is either true of false, e.g.:

the number 7 is prime

all multiples of 10 are also multiple of 5

2) involving one or more variables that becomes true of false whenever values are substituted for the variable, e.g.:

n is a multiple of 5

$x^2 < 20$

To prove mathematical statements, it is important to use clear, unambiguous language and valid logic.

Negating statements

The negation of a mathematical statement is the statement that is true precisely when the original statement is false, and vice-versa, e.g.:

the negation of the statement $x > 0$ is $x \leq 0$

As a general rule, the negation of a statement can be obtained by preceding the statement with the phrase "*it is not the case that*", e.g. if n represents an integer, then the negation of the statement "n *is an even number*" is "*it is not the case that* n *is an even number*" which is equivalent to saying that n is an odd number.

If P represents any statement, then the negation of P can be written as $\neg P$, $\sim P$ or simply $not\ P$

The negation of statements involving the words ‘and’ or ‘or’ can sometimes cause confusion. Consider negating the statement ‘either $x = 5$ or $x = 7$’. If it not the case that x is equal to 5 or 7, then it must be the case that $x \neq 5$ and $x \neq 7$.

Example: the negation of the statement ‘$x > 0$ *and* $x < 10$’ is ‘$x \leq 0$ *or* $x \geq 10$’

Generally:

The negation of ‘P *and* Q’ is ‘*not* P *or not* Q’

The negation of ‘P *or* Q’ is ‘*not* P *and not* Q’

These are known as **de Morgan’s laws.**

Example 1

Negate the following statements.

(a) n is divisible by 2 or n is divisible by 3

(b) $x > 0$ and $x < 5$.

Solution

(a) The opposite (negative of) ‘divisible’ is ‘not divisible’.
n is not divisible by 2 and n is not divisible by 3.
Or, n is divisible by neither 2 or 3.

(b) The opposite of ‘greater than’ is ‘not greater than’ or ‘less than or equal to’.
The opposite of ‘less than’ is ‘not less than’ or ‘greater than or equal to’.
Hence $x \leq 0$ or $x \geq 5$.

Statements involving quantifiers

The symbol $\in$ is used to mean “belonging to”, e.g.:

$x \in \mathbb{R}$	means	x belonging to $\mathbb{R}$ (the set of real numbers)
$x \in \mathbb{Z}$	means	x belonging to $\mathbb{Z}$ (the set of integers)

The symbol $\forall$ is used to mean “for all”, e.g.:

$\forall\, x \in \mathbb{N}$	means	for all x belonging to $\mathbb{N}$ (the set of natural numbers)
$\forall\, x \in \mathbb{Q}$	means	for all x belonging to $\mathbb{Q}$ (the set of rational numbers)
$\forall\, x \in \mathbb{R},\ x^2 \geq 0$	means	for all x belonging to $\mathbb{R}$, x^2 is positive.

The symbol $\exists$ is used to mean “there exists”.

$\exists\, n$ such that $n^2 = 9$	means	there exists n such that $n^2 = 9$

Example 2

Translate the following statements into everyday language. Also determine whether the statement is true or false, justifying your answer.

(a) $\exists\, x \in R$ such that $x^2 = \sqrt{x}$.

(b) $\forall$ integers n, the number $5n$ is even.

Solution

(a) There is at least one real number whose square is equal to its square root.
This is true as the number 1 satisfies this property.

(b) Multiplying any integer by 5 results in an even number.
This is false as, for example, $5 \times 3 = 15$, which is not even.

Example 3

Rewrite the following statements using the symbols $\forall$ and $\exists$. Also, state whether the statement is true or false, justifying your answer.

(a) The square root of any positive integer is less than or equal to the integer.

(b) There is at least one real number which, when squared, results in a smaller number.

Solution

(a) $\forall$ positive integers n, $\sqrt{n} \le n$.
This is a true statement as the square root of any number greater than or equal to 1 is less than or equal to the number itself.

(b) $\exists$ a real number x, such that $x^2 < x$.
This is true for $0 < x < 1$; for example, $0.5^2 = 0.25 < 0.5$.

Note that the symbols $\forall$ and $\exists$ may be used together in a single statement; however, the order in which they appear is important. As an example, consider the following two statements:

$\forall$ integers n, $\exists$ an integer m such that $n + m$ is a multiple of 5.

$\exists$ an integer n, such that $\forall$ integers m, $n + m$ is a multiple of 5.

The first statement is true as it says that 'for every integer, you can find another integer to add to it to give a sum that is a multiple of 5'.

The second statement is false as it says that 'there is a special integer that has the property that when you add any other integer to it, you always obtain a multiple of 5'.

> When the symbols $\forall$ and $\exists$ appear together in the same statement, the order in which they appear is important.

Examples and Counterexamples

In part **(a)** of Example **2**, it was claimed that the statement, '$\exists\, x \in R$ such that $x^2 = \sqrt{x}$' is true, and to justify this claim, a single example was provided of a real number whose square was equal to its square root (namely, $x = 1$). Providing a single example is always sufficient to prove that a 'there exists' statement is true.

In part **(b)** of Example **2**, it was claimed that the statement, '$\forall$ integers n, the number $5n$ is even' is false, and to justify this claim, a single example was provided of an integer which, when multiplied by 5, gives a number that is not even (namely, $n = 3$). An example, such as this, that demonstrates the falsehood of a statement is known as a counterexample. Providing a single counterexample is always sufficient to prove that a 'for all' statement is false.

If n is a multiple of 10, then n is an even number.

n is an even number if n is a multiple of 10.

n being a multiple of 10 is a *sufficient* condition to conclude that n is even.

n being even is *necessary* if n is a multiple of 10.

n is a multiple of 10 *implies that* n is an even number.

Negating statements involving quantifiers

Consider the negation of the statement '$\forall$ real numbers x, $x^2 \geq 0$'. If it is not the case that the square of every real number is greater than or equal to zero, then it must mean that there is at least one real number whose square is less than zero. Thus, the negation is '$\exists$ a real number x such that $x^2 < 0$'. Notice how the negation of a 'for all' statement is a 'there exists' statement. The reverse is also true. For example, the negation of the statement: '$\exists$ a real number x such that $3x = x$' is '$\forall$ real numbers x, $3x \neq x$'.

The negation of a 'for all' statement is a 'there exists' statement. Similarly, the negation of a 'there exists' statement is a 'for all' statement.

Example 4

Determine the negation of each of the following statements. Also state whether the original statement or the negation is true or false, justifying your answer.

(a) $\forall$ integers n, $2n$ is even.

(b) $\exists$ a real number x, such that $x^2 = -1$.

(c) $\exists$ an integer n, such that n is even and n is prime.

Solution

(a) $\exists$ an integer n such that $2n$ is odd.
Original statement is true as 2 multiplied by an integer is, by definition, even.

(b) $\forall$ real numbers x, $x^2 \neq -1$.
Negation is true, as the square of any real number is greater than or equal to zero.

(c) $\forall$ integers n, n is odd or n is not prime.
Original statement is true, as the number 2 is even and prime.

Conditional statements

Consider the following statement: if n is a multiple of 10, then n is an even number. This is an example of a conditional statement. A conditional statement (also known as an 'if-then' statement, or an 'implication') is one that asserts that *if* some condition holds, *then* it must be the case that some property is true. Conditional statements are so common in mathematics that there is a variety of ways to express them. The previous example, for instance, could be represented in any of the following ways:

If n is a multiple of 10, then n is an even number.

n is an even number if n is a multiple of 10.

n being a multiple of 10 is a *sufficient* condition to conclude that n is even.

n being even is *necessary* if n is a multiple of 10.

n is a multiple of 10 *implies that* n is an even number.

The implication symbol ⇒

The implication symbol, ⇒, is used to mean 'implies that', e.g:

n is a multiple of 10 ⇒ n is an even number

Each of the following means the same as $P \Rightarrow Q$:

If P, then Q

Q if P

P is a *sufficient* condition to conclude that Q

Q is *necessary* if P

P implies that Q

Example 5

Rewrite the following conditional statements using the implication symbol, ⇒.

(a) If n ends in a zero, then n is even.

(b) ∀ integers n, $n > 3$ is a sufficient condition to conclude that n is positive.

(c) $n > 3$ is necessary if n is greater than 4.

Solution

(a) 'If p, then q' can be written as $p \Rightarrow q$.
If n ends in a zero ⇒ n is even.

(b) 'p is a sufficient condition to conclude q' means the same as 'if p then q'.
$n > 3 \Rightarrow n$ positive.

(c) 'p is a necessary condition if q' means the same as 'if q, then p'.
$n > 4 \Rightarrow n > 3$

The converse of a conditional statement

The *converse* of a conditional statement is the statement obtained by swapping the statements on either side of the implication symbol. For example, consider the conditional statement previously introduced:

original: n is a multiple of 10 ⇒ n is an even number

converse: n is an even number ⇒ n is a multiple of 10

Notice that the converse is not saying the same thing as the original. The original statement is claiming that if a number is a multiple of 10, then it must be even (which is true). But the converse is claiming that if a number is even, then it must be a multiple of 10 (which is definitely not true).

The contrapositive of a conditional statement

The *contrapositive* of a conditional statement is the statement obtained by swapping the statements on either side of the implication symbol, and also negating both statements. Again, consider the conditional statement previously introduced:

original: n is a multiple of 10 $\Rightarrow$ n is an even number

contrapositive: n is not an even number $\Rightarrow$ n is not multiple of 10

Notice that the contrapositive *is* actually true, just like the original. In a sense, it is saying the exact same thing as the original statement.

As a classic illustrative real-life example, the contrapositive of the statement 'if an animal is a poodle, then it is a dog' is 'if an animal is not a dog, then it is not a poodle'. Notice, again, how the original and the contrapositive statements are essentially saying the same thing.

Now consider the *negation* of the statement 'n is a multiple of 10 $\Rightarrow$ n is an even number'. Remember that that this statement is essentially saying that for every integer that is a multiple of 10, this integer must also be even.

If this were not the case, it would mean that there must exist some integer that is a multiple of 10 but is not even. In general, the negation of a conditional statement of the form $P \Rightarrow Q$ that involves some variable is 'there exists some value of the variable for which P is true, but Q is false'.

Using the real-life example from earlier, the negation of 'if an animal is a poodle, then it is a dog' would be 'there exists some animal that is a poodle, but not a dog'.

Notice that the negation of a conditional statement is different from both the converse, and the contrapositive.

For a statement of the form $P \Rightarrow Q$ that involves some variable:

The converse is the statement $Q \Rightarrow P$;

The contrapositive is the statement *not* $Q \Rightarrow$ *not* P;

The negation is the statement 'there exists some value of the variable for which P is true, but Q is false'.

The contrapositive of a conditional statement essentially says the same thing as the original statement, and thus, will be true whenever the original statement is true.

Example 6

Write the converse, the contrapositive, and the negation of the following conditional statement.

If n is a perfect square, then n is divisible by 3.

Determine whether each of the original, converse, contrapositive, and negation are true or false, justifying your answer.

Solution

Is the original statement true or false? The original statement is false as, for example, 16 is a perfect square but it is not divisible by 3.

The converse of $P \Rightarrow Q$ is $Q \Rightarrow P$: Converse is 'if n is divisible by 3 then n is a perfect square'. For example, 12 is divisible by 3 but it is not a perfect square.

The contrapositive of a statement of the form $P \Rightarrow Q$ is *not* $Q \Rightarrow$ *not* P: Contrapositive is 'if n is not divisible by 3 then n is not a perfect square'. Like the original statement, this statement is false.

The negation of a statement of the form $P \Rightarrow Q$ is 'there exists some value of the variable for which P is true, but Q is false': Negation is 'there exists an integer n with the property that n is a perfect square, but n is not divisible by 3'. This statement must be true as the original statement was false. $n = 16$ has this property.

Notice that in the previous worked example, a counterexample was provided to justify the claim that the statement 'If n is a perfect square, then n is divisible by 3' is false. This should make sense, as conditional statements such as these are making a claim about all possible values of a variable that satisfy some condition, and are therefore similar to 'for all' statements (which are proved false by providing a counterexample). Incidentally, this statement could be rephrased as 'For all perfect square integers n, n is divisible by 3'.

Logically equivalent statements – The symbol $\Leftrightarrow$

Recall that the conditional statement 'n is a multiple of 10 $\Rightarrow$ n is an even number' is true, however, its converse 'n is an even number $\Rightarrow$ n is a multiple of 10' is not. Sometimes, however, a conditional statement and its converse are both true. As an example, notice that if $x = 5$, then $2x = 10$, and conversely, if $2x = 10$, then $x = 5$. This means that for the two statements, $x = 5$ and $2x = 10$, whenever one is of these is true, the other must be. Such statements are said to be *logically equivalent*.

Two statements are logically equivalent if whenever one is true, the other must be true.

There are a variety of ways to represent the fact that $x = 5$ and $2x = 10$ are logically equivalent:

$x = 5$ is *necessary* and *sufficient* for $2x = 10$

$x = 5$ *if and only if* $2x = 10$

$x = 5 \Rightarrow 2x = 10$ and $2x = 10 \Rightarrow x = 5$

The symbol $\Leftrightarrow$ is used to denote logical equivalence, e.g.:

$x = 5 \Leftrightarrow 2x = 10$

Each of the following can be used to express the fact that P and Q are logically equivalent:

P is *necessary* and *sufficient* for Q

P *if and only if* Q

$P \Rightarrow Q$ and $Q \Rightarrow P$

$P \Leftrightarrow Q$

Example 7

Rewrite the following statement using the logical equivalence symbol, $\Leftrightarrow$:

For n to be divisible by 5, it is both necessary and sufficient that n end in either 0 or 5.

Solution

'P is a necessary and sufficient condition for Q' means that P and Q are logically equivalent: n is divisible by 5 $\Leftrightarrow$ n ends in 0 or 5.