DERIVATIVE OF THE PRODUCT OF TWO FUNCTIONS (the “product rule”)

If $f(x) = u(x) \times v(x)$ then $f'(x) = u(x)\, v'(x) + v(x)\, u'(x)$

or noted in a simplified way: $(uv)' = uv' + vu'$

Proof:

Let $f(x) = u(x) \times v(x)$

By definition: $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$f'(x) = \lim_{h \to 0} \left\{ \frac{u(x+h)v(x+h) - u(x)v(x)}{h} \right\}$$

We subtract and add the term $u(x+h)v(x)$ to the numerator.

$$f'(x) = \lim_{h \to 0} \left\{ \frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h} \right\}$$

$$f'(x) = \lim_{h \to 0} \left\{ \frac{u(x+h)[v(x+h) - v(x)] + v(x)[u(x+h) - u(x)]}{h} \right\}$$

$$f'(x) = \lim_{h \to 0} \left\{ u(x+h) \frac{[v(x+h) - v(x)]}{h} + v(x) \frac{[u(x+h) - u(x)]}{h} \right\}$$

$$f'(x) = \lim_{h \to 0} \left\{ u(x+h) \frac{[v(x+h) - v(x)]}{h} \right\} + \lim_{h \to 0} \left\{ v(x) \frac{[u(x+h) - u(x)]}{h} \right\}$$

But $\lim\limits_{h \to 0} u(x+h) = u(x)$ so:

$$f'(x) = u(x) \times \lim_{h \to 0} \left[\frac{v(x+h) - v(x)}{h} \right] + v(x) \times \lim_{h \to 0} \left[\frac{u(x+h) - u(x)}{h} \right]$$

Therefore: $f'(x) = u(x)\, v'(x) + v(x)\, u'(x)$

In simplified notation: $(uv)' = uv' + vu'$ referred to as the “*product rule*”

Example: if $f(x) = (4x^5 + 2x)(2\sqrt{x} - x^2)$ let: $u(x) = 4x^5 + 2x$ and $v(x) = 2\sqrt{x} - x^2$

then : $u'(x) = 20x^4 + 2$ and $v'(x) = 2 \times \frac{1}{2} x^{\frac{1}{2}-1} - 2x = x^{-\frac{1}{2}} - 2x = \frac{1}{\sqrt{x}} - 2x$

therefore $f'(x) = (4x^5 + 2x)\left(\frac{1}{\sqrt{x}} - 2x\right) + (2\sqrt{x} - x^2)(20x^4 + 2)$