DISCRETE DISTRIBUTIONS IN PRACTICAL SITUATIONS

1 A die, labelled 1 to 6, is rolled until the total of the scores is 4 or greater. Answer each of the following, giving all answers as exact values, in simplest fraction form.

(a) Find the probability distribution of the number of rolls X required to achieve this total.

(b) Find the expected number of rolls required.

(c) Find the variance for the number of rolls required.

a) The maximum number of rolls is 4 (four ones).

$P(X=1) = 1/2$

$P(X=2) = \frac{1}{6} \times \frac{4}{6} + \frac{1}{6} \times \frac{5}{6} + \frac{1}{6} \times \frac{6}{6} = \frac{15}{36} = \frac{5}{12}$

$P(X=4) = \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{6}{6} = \frac{1}{216}$

$P(X=3) = 1 - P(X=1) - P(X=2) - P(X=4) = 1 - \frac{1}{2} - \frac{5}{12} - \frac{1}{216} = \frac{17}{216}$

x_i	1	2	3	4
$P(X=x_i)$	$1/2$	$5/12$	$17/216$	$1/216$

is the probability distribution table

b) $E(X) = 1 \times \frac{1}{2} + 2 \times \frac{5}{12} + 3 \times \frac{17}{216} + 4 \times \frac{1}{216} = \frac{343}{216}$

or $1\ 127/216$

c) $Var(X) = E(X^2) - [E(X)]^2$

$E(X^2) = 1^2 \times \frac{1}{2} + 2^2 \times \frac{5}{12} + 3^2 \times \frac{17}{216} + 4^2 \times \frac{1}{216}$

$E(X^2) = \frac{637}{216}$

so $Var(X) = \frac{637}{216} - \left[\frac{343}{216}\right]^2 = \frac{19943}{46656}$

prime numbers
2, 3, 5

2 One of the games at the local sporting club's 'Vegas Night' involves rolling a standard six-sided die. If a non-prime number is shown, there is no game charge and the player wins the number of dollars shown on the face of the die. If a prime number is shown, the cost of playing is the number of dollars shown on the face of the die. Let Z stand for the number of dollars received by the player.

(a) Draw a table to show the probability distribution of the variable.
(b) Is this best described as a uniform or a non-uniform distribution?
(c) Find the value of $E(Z)$.
(d) Is this game fair? If not, is it biased in favour of the operator or the player?

a)

x_i	1	2	3	4	5	6
$P(X=x_i)$	1/6	1/6	1/6	1/6	1/6	1/6
Z	1	−2	−3	4	−5	6

b) This is a uniform distribution

c) $E(Z) = 1 \times \frac{1}{6} + (-2) \times \frac{1}{6} + (-3) \times \frac{1}{6} + 4 \times \frac{1}{6} + (-5) \times \frac{1}{6} + 6 \times \frac{1}{6}$

So $E(Z) = 1/6$

d) If the game was fair, then $E(Z)$ would be equal to 0
But in fact $E(Z) = \frac{1}{6} > 0$, so it is advantageous to the player. The game is biaised in favour of the player.

4 For the discrete random variable X, the probability distribution is given by:

$$P(X = x) = \begin{cases} kx, & x = 1, 2, 3, 4 \\ k(9-x), & x = 5, 6, 7, 8 \end{cases}$$

(a) Find the value of k.

(b) Complete the following table to show the probability distribution of X.

x	1	2	3	4	5	6	7	8
$P(X = x)$	$1/20$	$2/20$	$3/20$	$4/20$	$4/20$	$3/20$	$2/20$	$1/20$

(c) Find the value of $E(X)$. **(d)** Find the value of $\mathrm{Var}(X)$.

a) $\underbrace{k + 2k + 3k + 4k + (9-5)k + (9-6)k + (9-7)k + (9-8k)}$ must be equal to 1

So $20k = 1$ so $k = 1/20$

c) $E(X) = 1 \times \frac{1}{20} + 2 \times \frac{2}{20} + 3 \times \frac{3}{20} + 4 \times \frac{4}{20} + 5 \times \frac{4}{20} + 6 \times \frac{3}{20} + 7 \times \frac{2}{20} = \frac{90}{20} = 4.5$

d) $\mathrm{Var}(X) = E(X^2) - [E(X)]^2$

$E(X^2) = 1^2 \times \frac{1}{20} + 2^2 \times \frac{2}{20} + 3^2 \times \frac{3}{20} + 4^2 \times \frac{4}{20} + 5^2 \times \frac{4}{20} + 6^2 \times \frac{3}{20} + 7^2 \times \frac{2}{20} + 8^2 \times \frac{1}{20}$

So $E(X^2) = \frac{47}{2} = 23.5$

$\mathrm{Var}(X) = 23.5 - [4.5]^2 = 3.25$

6 Tomino has written the following as his answer for the probability distribution of the random variable X:

$$P(X = x) = \begin{cases} \frac{4-x}{5} & \text{for } x = 1, 2, 3, 4, 5 \\ 0 & \text{otherwise} \end{cases}$$

Explain to Tomino why this cannot be correct.

For $x = 5$ $P(X=5) = \frac{4-5}{5} = -\frac{1}{5}$

a probability cannot be negative, so that's impossible

7 The discrete random variable X has the probability distribution shown in the following table.

x	1	2	3
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

(a) Find $E(X)$.
(b) Find $\mathrm{Var}(X)$.
(c) A second random variable Y has the same distribution as X, and the two variables are independent. Draw a table to show the probability distribution of $X + Y$.
(d) Find $E(X + Y)$.
(e) Find $\mathrm{Var}(X + Y)$.
(f) How is the value of $E(X + Y)$ related to the values of $E(X)$ and $E(Y)$?
(g) How is the value of $\mathrm{Var}(X + Y)$ related to the values of $\mathrm{Var}(X)$ and $\mathrm{Var}(Y)$?

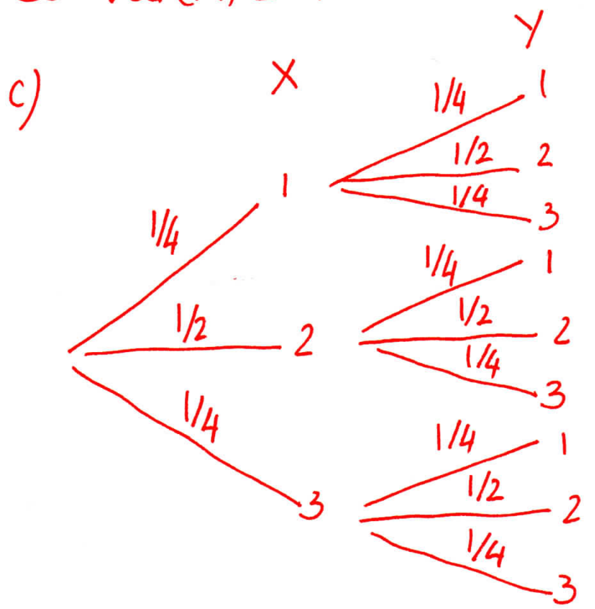
a) $E(X) = 1 \times \frac{1}{4} + 2 \times \frac{1}{2} + 3 \times \frac{1}{4} = 2$

b) $\mathrm{Var}(X) = E(X^2) - [E(X)]^2$

$E(X^2) = 1^2 \times \frac{1}{4} + 2^2 \times \frac{1}{2} + 3^2 \times \frac{1}{4} = \frac{9}{2} = 4.5$

So $\mathrm{Var}(X) = 4.5 - 2^2 = 0.5$

c)



$P(2) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$
$P(3) = 1/4 \times 1/2 = 1/8$
$P(4) = 1/4 \times 1/4 = 1/16$
$P(3) = 1/2 \times 1/4 = 1/8$
$P(4) = 1/2 \times 1/2 = 1/4$
$P(5) = 1/2 \times 1/4 = 1/8$
$P(4) = 1/4 \times 1/4 = 1/16$
$P(5) = 1/4 \times 1/2 = 1/8$
$P(6) = 1/4 \times 1/4 = 1/16$

x_i	2	3	4	5	6
$P(X+Y=x_i)$	$\frac{1}{16}$	$\frac{1}{8}+\frac{1}{8}=\frac{1}{4}$	$\frac{1}{16}+\frac{1}{4}+\frac{1}{16}$ $= 3/8$	$\frac{1}{8}+\frac{1}{8}=\frac{1}{4}$	$\frac{1}{16}$

d) $E(X+Y) = 2\times\frac{1}{16} + 3\times\frac{1}{4} + 4\times\frac{3}{8} + 5\times\frac{1}{4} + 6\times\frac{1}{16} = 4$

e) $Var(X+Y) = E(X+Y)^2 - [E(X+Y)]^2$

$E(X+Y)^2 = 2^2\times\frac{1}{16} + 3^2\times\frac{1}{4} + 4^2\times\frac{3}{8} + 5^2\times\frac{1}{4} + 6^2\times\frac{1}{16} = 17$

So $Var(X+Y) = 17 - 4^2 = 1$

f) from a) $E(X) = 2$ and $E(Y) = 2$ (same distribution than X)

whereas $E(X+Y) = 4$

So in that case $E(X+Y) = E(X) + E(Y)$.

$E(X+Y)$ is the sum of $E(X)$ and of $E(Y)$

g) from b) $Var(X) = 0.5$ and $Var(Y) = 0.5$ (same distribution than X)

whereas $Var(X+Y) = 1$.

So in that case, $Var(X+Y) = Var(X) + Var(Y)$

$Var(X+Y)$ is the sum of $Var(X)$ and of $Var(Y)$.

9 Stephan has made a game in which the probability of randomly picking a number from 0 to 5 is given by the probability distribution shown in the following table. Answer each of the following, giving all answers correct to 3 decimal places.

X	0	1	2	3	4	5
$P(X=x)$	0.002	0.076	0.293	0.268	a	0.098

(a) Calculate the expected value for this random variable.

(b) Leanne made a game similar to Stephan's, but the probability of randomly picking a number from 0 to 5 is given by the following probability distribution.

Y	0	1	2	3	4	5
$P(Y=y)$	0.005	0.029	0.047	0.219	0.386	0.314

Calculate the expected value for this random variable.

(c) If one value in Stephan's game and one value in Leanne's game are chosen at random, calculate the probability that:

(i) they are the same value **(ii)** they are different **(iii)** their sum is greater than 8.

$0.002 + 0.076 + 0.293 + 0.268 + a + 0.098 = 1$

$\Leftrightarrow a = 0.263$

a) $E(X) = 0 \times 0.002 + 1 \times 0.076 + 2 \times 0.293 + 3 \times 0.268 + 4 \times 0.263 + 5 \times 0.098$
$= 3.008$

b) $E(Y) = 0 \times 0.005 + 1 \times 0.029 + 2 \times 0.047 + 3 \times 0.219 +$
$4 \times 0.386 + 5 \times 0.314 = 3.894$

c) i) $P = 0.002 \times 0.005 + 0.076 \times 0.029 + 0.293 \times 0.047 + 0.268 \times 0.219$
$+ 0.263 \times 0.386 + 0.098 \times 0.314 = 0.206967 \simeq 0.21$

ii) P(they are the same) + P(they are different) = 1

so P(they are different) $= 1 - 0.206967 \simeq 0.793$

iii) So their sum must be 9 or 10

P(sum is 9 or 10) $= 0.263 \times 0.314 + 0.098 \times 0.386 + 0.098 \times 0.314$
$= 0.151182 \simeq 0.152$