FIRST PRINCIPLE OF DIFFERENTIATION

2 Use the result $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ to find:

(a) $f'(-2)$ when $f(x)=x^2$ **(b)** $f'(-1)$ when $f(x)=x^3$

a) for $f(x)=x^2$ $f'(x)=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}=\lim_{h\to 0}\frac{[x^2+2xh+h^2-x^2]}{h}$

$f'(x)=\lim_{h\to 0}\frac{2xh+h^2}{h}=\lim_{h\to 0} 2x+h = 2x$

So $f'(-2)=2\times -2=-4$

the slope of the function f at $x=-2$ is equal to -4.

b) $f'(x)=\lim_{h\to 0}\frac{(x+h)^3-x^3}{h}=\lim_{h\to 0}\frac{[x^3+3x^2h+3xh^2+h^3-x^3]}{h}$

$f'(x)=\lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}$

$f'(x)=\lim_{h\to 0} 3x^2+3xh+h^2$

$f'(x)=3x^2$

So $f'(-1)=3\times(-1)^2=3$

The gradient of the curve $y=x^3$ at $x=-1$ is equal to 3.

FIRST PRINCIPLE OF DIFFERENTIATION

3 $P(1,1)$ and $Q(2,8)$ are points on the curve $f(x)=x^3$. Indicate whether each statement is correct or incorrect.

(a) Gradient of $PQ=7$ **(b)** $f'(2)=\lim\limits_{x\to 1}\dfrac{x^3-8}{x-2}$ **(c)** $f'(1)=\lim\limits_{x\to 1}\dfrac{x^3-1}{x-1}$ **(d)** $f'(x)=3x^2$

Gradient of $PQ = \dfrac{\text{Rise}}{\text{Run}} = \dfrac{8-1}{2-1} = \dfrac{7}{1} = 7$ so a) is true.

$f'(x) = \lim\limits_{h\to 0} \dfrac{(x+h)^3 - x^3}{h} = 3x^2$ as demonstrated at Question 2 b)

so d) is true.

So $f'(2) = 3\times 2^2 = 3\times 4 = 12$

whereas $\lim\limits_{x\to 1} \dfrac{x^3-8}{x-2} = \lim\limits_{x\to 1} \dfrac{(x-2)(x^2+2x+4)}{x-2}$

$= \lim\limits_{x\to 1} x^2+2x+4$

$= 1^2 + 2\times 1 + 4 = 7$

so $f'(2) \neq \lim\limits_{x\to 1} \dfrac{x^3-8}{x-2}$ b) is not true.

$f'(1) = 3\times 1^2 = 3$

whereas $\lim\limits_{x\to 1} \dfrac{x^3-1}{x-1} = \lim\limits_{x\to 1} \dfrac{(x-1)(x^2+x+1)}{x-1}$

$= \lim\limits_{x\to 1} x^2+x+1$

$= 1^2+1+1 = 3$

so $f'(1) = \lim\limits_{x\to 1} \dfrac{x^3-1}{x-1}$ is true. c) is true.

FIRST PRINCIPLE OF DIFFERENTIATION

5 For the function $f(x) = 2x^2 - 4x$, find the following:

(a) $\lim\limits_{h\to 0} \dfrac{f(3+h) - f(3)}{h}$ **(b)** $\lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

Interpret your results geometrically.

a) $\lim\limits_{h\to 0} \dfrac{f(3+h) - f(3)}{h} = \lim\limits_{h\to 0} \dfrac{[2(3+h)^2 - 4(3+h)] - [2\times 3^2 - 4\times 3]}{h}$

$= \lim\limits_{h\to 0} \dfrac{[\cancel{18} + 12h + 2h^2 - \cancel{12} - 4h - \cancel{18} + \cancel{12}]}{h}$

$= \lim\limits_{h\to 0} \dfrac{2h^2 + 8h}{h}$

$= \lim\limits_{h\to 0} 2h + 8 = 8$

So the gradient of f at $x=3$ is 8.

b) $\lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{[2(x+h)^2 - 4(x+h)] - [2x^2 - 4x]}{h}$

$= \lim\limits_{h\to 0} \dfrac{[\cancel{2x^2} + 4xh + 2h^2 - \cancel{4x} - 4h - \cancel{2x^2} + \cancel{4x}]}{h}$

$= \lim\limits_{h\to 0} \dfrac{[4xh + 2h^2 - 4h]}{h}$

$= \lim\limits_{h\to 0} 4x + 2h - 4$

$= 4x - 4$

So $f'(x) = 4x - 4$

6 Find $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ for the following:

(a) $f(x)=4x^2-1$

(c) $f(x)=x^3-2x^2$

a) $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0}\dfrac{[4(x+h)^2-1]-[4x^2-1]}{h}$

$= \lim\limits_{h\to 0}\dfrac{[\cancel{4x^2}+8xh+4h^2\cancel{-1}-\cancel{4x^2}\cancel{+1}]}{h}$

$= \lim\limits_{h\to 0}\dfrac{[8xh+4h^2]}{h}$

$= \lim\limits_{h\to 0} 8x+4h = 8x$ So $f'(x)=8x$

b) $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0}\dfrac{[(x+h)^3-2(x+h)^2]-[x^3-2x^2]}{h}$

$= \lim\limits_{h\to 0}\dfrac{[\cancel{x^3}+3x^2h+3xh^2+h^3-\cancel{2x^2}-4xh-2h^2\cancel{-x^3}\cancel{+2x^2}]}{h}$

$= \lim\limits_{h\to 0}\dfrac{3x^2h+3xh^2-4xh+h^3-2h^2}{h}$

$= \lim\limits_{h\to 0} 3x^2+3xh-4x+h^2-2h$

$= 3x^2-4x$

So $f'(x)=3x^2-4x$

d) $f(x) = x^3 + 4x + 5$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^3 + 4(x+h) + 5] - [x^3 + 4x + 5]}{h}$$

$$= \lim_{h \to 0} \frac{[\cancel{x^3} + 3x^2h + 3xh^2 + h^3 + \cancel{4x} + 4h + \cancel{5} - \cancel{x^3} - \cancel{4x} - \cancel{5}]}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 + 4h}{h}$$

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2 + 4$$

$$= 3x^2 + 4$$

Hence $f'(x) = 3x^2 + 4$

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e) $f(x) = x^4$

You will need to use the expansion $(x + h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[x+h]^4 - x^4}{h}$$

$$= \lim_{h \to 0} \frac{[\cancel{x^4} + 4x^3h + 6x^2h^2 + 4xh^3 + h^4] - \cancel{x^4}}{h}$$

$$= \lim_{h \to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h}$$

$$= \lim_{h \to 0} 4x^3 + 6x^2h + 4xh^2 + h^3$$

$$= 4x^3$$

Hence $f'(x) = 4x^3$