THE CHAIN RULE

1 Use the chain rule to differentiate:

(a) $y=(x^2-4)^5$ (b) $f(x)=\sqrt{5x-1}$ (c) $y=(x^3-3x)^4$

a) $f(x)=[u(x)]^5$ with $u(x)=x^2-4$ so $u'(x)=2x$

$f'(x)=5[u(x)]^4\times u'(x)=5[x^2-4]^4\times 2x=10x(x^2-4)^4$

b) $f(x)=\sqrt{u(x)}=(u(x))^{1/2}$ with $u'(x)=5$

So $f'(x)=\frac{1}{2}[u(x)]^{1/2-1}\times u'(x)=\frac{1}{2[u(x)]^{1/2}}\times u'(x)$

$f'(x)=\frac{1\times 5}{2\sqrt{5x-1}}=\frac{5}{2\sqrt{5x-1}}$

c) $f(x)=[u(x)]^4$ with $u(x)=x^3-3x$ so $u'(x)=3x^2-3$

$f'(x)=4[u(x)]^3\times u'(x)$

$f'(x)=4(x^3-3x)^3\times(3x^2-3)$

$f'(x)=12[x^3-3x]^3(x^2-1)$

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(g) $f(x)=\sqrt{x^2-2x}$ **(h)** $f(t)=(t^2+4)^{-2}$ **(i)** $y=\sqrt{25-x^2}$

g) $f(x)=[u(x)]^{1/2}$ with $u(x)=x^2-2x$ so $u'(x)=2x-2$

$$f'(x)=\frac{1}{2}[u(x)]^{1/2-1}\times u'(x)=\frac{u'(x)}{2[u(x)]^{1/2}}$$

$$f'(x)=\frac{2(x-1)}{2\sqrt{x^2-2x}}=\frac{x-1}{\sqrt{x^2-2x}}$$

h) $f(t)=[u(t)]^{-2}$ with $u(t)=t^2+4$ so $u'(t)=2t$

$$f'(t)=-2[u(t)]^{-2-1}\times u'(t)=-2[u(t)]^{-3}\times u'(t)$$

$$f'(t)=\frac{-2}{(t^2+4)^3}\times 2t=\frac{-4t}{(t^2+4)^3}$$

i) $f(x)=\sqrt{u(x)}=[u(x)]^{1/2}$ with $u(x)=25-x^2$ so $u'(x)=-2x$

$$f'(x)=\frac{1}{2}[u(x)]^{1/2-1}\times u'(x)=\frac{u'(x)}{2[u(x)]^{1/2}}=\frac{u'(x)}{2\sqrt{u(x)}}$$

$$f'(x)=\frac{-2x}{2\sqrt{25-x^2}}=-\frac{x}{\sqrt{25-x^2}}$$

3 Find the derivative of each function.

(a) $y=\sqrt{x^2-4}$ **(b)** $f(x)=(x^2+1)^{\frac{1}{2}}$ **(c)** $y=(1+2x)^{-1}$

a) $f(x)=\sqrt{u(x)}=[u(x)]^{1/2}$ with $u(x)=x^2-4$ so $u'(x)=2x$

$f'(x)=\frac{1}{2}[u(x)]^{1/2-1}\times u'(x)=\frac{u'(x)}{2[u(x)]^{1/2}}=\frac{u'(x)}{2\sqrt{u(x)}}$

$f'(x)=\frac{2x}{2\sqrt{x^2-4}}=\frac{x}{\sqrt{x^2-4}}$

b) $f(x)=[u(x)]^{1/2}$ with $u(x)=x^2+1$ so $u'(x)=2x$

$f'(x)=\frac{u'(x)}{2\sqrt{u(x)}}=\frac{2x}{2\sqrt{x^2+1}}=\frac{x}{\sqrt{x^2+1}}$

c) $f(x)=[u(x)]^{-1}$ with $u(x)=1+2x$ so $u'(x)=2$

$f'(x)=(-1)\times[u(x)]^{-1-1}\times u'(x)=-\frac{u'(x)}{[u(x)]^{-2}}$

so $f'(x)=-\frac{2}{[1+2x]^2}$

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3 Find the derivative of each function.

(g) $f(x) = (3x^2 - 2x - 1)^4$ **(h)** $f(t) = (t^2 + 4)^{-2}$ **(i)** $y = \left(x - \frac{1}{x}\right)^4$

g) $f(x) = [u(x)]^4$ with $u(x) = 3x^2 - 2x - 1$ so $u'(x) = 6x - 2$

$f'(x) = 4[u(x)]^3 \times u'(x) = 4[3x^2 - 2x - 1]^3 \times (6x - 2)$

$f'(x) = 8[3x^2 - 2x - 1]^3 (3x - 1)$

h) $f(t) = [t^2 + 4]^{-2} = [u(t)]^{-2}$ with $u(t) = t^2 + 4$ so $u'(t) = 2t$

$f'(t) = -2[u(t)]^{-2-1} \times u'(t) = -\dfrac{2u'(t)}{[u(t)]^3}$

$f'(t) = -\dfrac{2 \times 2t}{[t^2 + 4]^3} = -\dfrac{4t}{[t^2 + 4]^3}$

i) $f(x) = [u(x)]^4$ with $u(x) = x - \dfrac{1}{x} = x - x^{-1}$ so $u'(x) = 1 + \dfrac{1}{x^2}$

$f'(x) = 4[u(x)]^3 \times u'(x) = 4\left[x - \dfrac{1}{x}\right]^3 \times \left(1 + \dfrac{1}{x^2}\right)$

4 For $g(x)=x^2+5x+\sqrt[3]{x^2-4}$, indicate whether each statement is correct or incorrect.

(a) $g'(x)=\frac{2x}{3}(x^2-4)^{-\frac{2}{3}}$

(b) $g'(x)=4x$

(c) $g'(x)=2x+5+\frac{2x}{3}(x^2-4)^{-\frac{2}{3}}$

(d) $g'(x)=2x+5+\frac{2x}{3(x^2-4)^{\frac{2}{3}}}$

$g(x)=x^2+5x+f(x)$ with $f(x)=(x^2-4)^{1/3}$

We find $f'(x)$ $f(x)=[u(x)]^{1/3}$ with $u(x)=x^2-4$ so $u'(x)=2x$

$$f'(x)=\frac{1}{3}[u(x)]^{1/3-1}\times u'(x)=\frac{1}{3}[u(x)]^{-2/3}\times u'(x)$$

$$f'(x)=\frac{u'(x)}{3[u(x)]^{2/3}}=\frac{2x}{3[x^2-4]^{2/3}}$$

So $g'(x)=2x+5+\frac{2x}{3[x^2-4]^{2/3}}$

CORRECT = c) d)

INCORRECT = a) b)

5 Find the derivative of each function.

(a) $y=(x-3)(3x+4)^6$ **(b)** $f(x)=x^2\sqrt{1-x^2}$ **(c)** $h(t)=t^3+(4-t)^4$

a) $f(x)=(x-3)(3x+4)^6 = g(x)\times h(x)$

$f'(x)=g'(x)\,h(x)+g(x)\,h'(x)$ (Product rule)

$g(x)=x-3$ $g'(x)=1$

$h(x)=(3x+4)^6$ $h'(x)=6(3x+4)^5\times 3$ (chain rule)

So $f'(x)=1\times(3x+4)^6+(x-3)\times[18(3x+4)^5]$

$f'(x)=[3x+4]^5\,[(3x+4)+18(x-3)]=[3x+4]^5\,[21x-50]$

b) $f(x)=g(x)\,h(x)$ ... so we use the product rule.

$g(x)=x^2$ $g'(x)=2x$

$h(x)=(1-x^2)^{1/2}$ $h'(x)=\frac{1}{2}(1-x^2)^{-1/2}\times(-2x)$ (Chain rule)

$h'(x)=-\frac{x}{\sqrt{1-x^2}}$

$f'(x)=2x\sqrt{1-x^2}-\frac{x^3}{\sqrt{1-x^2}}=\frac{2x(1-x^2)-x^3}{\sqrt{1-x^2}}=\frac{2x-3x^3}{\sqrt{1-x^2}}$

c) $h'(t)=3t^2+[(4-t)^4]'$

$h'(t)=3t^2+4(4-t)^3\times(-1)$

So $h'(t)=3t^2-4(4-t)^3$