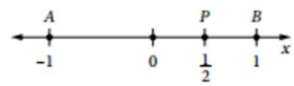
VELOCITY AND ACCELERATION AS FUNCTIONS OF x

The expressions for displacement $x(t)$, velocity $v(t)=\dot{x}(t)$ and acceleration $a(t)=\ddot{x}(t)=\dot{v}(t)$ all clearly define functions of time t. This makes sense because an ordinary particle cannot be in two different places at the same time: for each value of t there corresponds only one value of $x(t)$ that satisfies the function. Similarly, $v(t)$ and $a(t)$ define functions because a particle cannot be moving in two different ways (that is, cannot have more than one velocity and one acceleration) at any given time.

However, sometimes it is still useful to express acceleration in terms of the displacement x or in terms of the velocity v, rather than in terms of time t. In this case you must be careful, because equations relating a, v and x will not necessarily define functions.

Consider a particle that moves so that its displacement at time $t \geq 0$ is given by $x(t) = \sin t$. This means that the particle is moving back and forth between the limits $x = \pm 1$. (This is an example of simple harmonic motion, an important kind of motion that is explored in more detail later in this chapter.)

The particle is at $x=\frac{1}{2}$ whenever $\sin t=\frac{1}{2}$, i.e. when $t=\frac{\pi}{6}, \pi-\frac{\pi}{6}, 2\pi+\frac{\pi}{6}, \ldots$ so that $x(t)$ defines a many-to-one function of time. Similarly, $v(t)=\dot{x}(t)=\cos t$ defines a many-to-one function of time.



Note that $v^2=\cos^2 t=1-\sin^2 t=1-x^2$. Hence the equation $v^2=1-x^2$ expresses v in terms of x. However, this equation is not a function of x, because (for example) at $x=\frac{1}{2}$ you may have $v=\pm\frac{\sqrt{3}}{2}$. The point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ corresponds to $t=\frac{\pi}{6}$, while $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ corresponds to $t=\frac{5\pi}{6}$. There is an infinite number of values of t for each value of x. As the particle moves to and fro between A and B, its velocity has two possible values at any point P in between, depending on whether the particle is moving towards A or towards B at the time. At the extremes A and B, the velocity is zero, so at these points you can say the particle is instantaneously at rest.

Remember that differential calculus needs to be applied to functions. In this context, $\frac{dv}{dx}$ can only have a useful meaning if $v=v(x)$ defines a function. Hence if you want to differentiate the equation $v^2=1-x^2$ you must restrict v to be either positive or negative. This means that in the equation $v^2=1-x^2$ you consider v to apply only to velocities as the particle moves from A to B, i.e. $v>0$, or only from B to A, i.e. $v<0$.

If $v(x)$ specifies a velocity function of x according to one of these restrictions, then you can use the chain rule of differentiation to calculate acceleration:

$$\begin{aligned}\frac{dv}{dt} &= \frac{dv}{dx}\times\frac{dx}{dt} \\ &= v\frac{dv}{dx} \quad \text{as } v=\frac{dx}{dt} \\ &= \frac{d}{dv}\left(\frac{1}{2}v^2\right)\times\frac{dv}{dx} \\ &= \frac{d}{dx}\left(\frac{1}{2}v^2\right)\end{aligned}$$

Hence acceleration may be expressed in any of the forms $\frac{dv}{dt}, \frac{d^2x}{dt^2}, v\frac{dv}{dx}$ or $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$.

The form to use in a particular problem will depend on the form of the equation that defines acceleration:

- Given $a=f(t)$, use $\frac{dv}{dt}$ or $\frac{d^2x}{dt^2}$
- Given $a=g(x)$, use $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$
- Given $a=h(v)$, use $\begin{cases}\frac{dv}{dt} & \text{if initial conditions are values for } t \text{ and } v \\ v\frac{dv}{dx} & \text{if initial conditions are values for } x \text{ and } v\end{cases}$

It is customary to write derivatives with respect to time using dots above the dependent variable, e.g. $\dot{x}=\frac{dx}{dt}, \ddot{x}=\frac{d^2x}{dt^2}, \dot{v}=\frac{dv}{dt}$.

Example 1

A particle moves in a straight line so that at time t its displacement from a fixed origin is x and its velocity is v. If the acceleration is $3-2x$, find v in terms of x given that $v=2$ when $x=1$.

Solution

$\ddot{x}=3-2x$ gives the acceleration as a function of x and the initial conditions are v and x, so to find v as a function of x:

Use $\ddot{x}=\frac{d}{dx}\left(\frac{1}{2}v^2\right)$: $\quad \frac{d}{dx}\left(\frac{1}{2}v^2\right)=3-2x$

Integrate with respect to x: $\quad \frac{1}{2}v^2=\int(3-2x)dx$

$$\frac{1}{2}v^2=3x-x^2+C$$

At $x=1$, $v=2$: $\quad 2=3-1+C$

$$C=0$$

$$\therefore \quad \frac{1}{2}v^2=3x-x^2$$

$$v^2=6x-2x^2$$

$$v=\pm\sqrt{6x-2x^2}$$

The conditions include $x=1$, $v=2$, which means that v is positive when $x=1$.
To satisfy the initial conditions you must take the positive square root.

$$\therefore \quad v=\sqrt{6x-2x^2}$$

It is worth noting the domain of the velocity function. For v to exist, $6x-2x^2\geq 0$, so $0\leq x\leq 3$: this means that the motion exists only for $0\leq x\leq 3$.

Example 3

The velocity of a particle is $v=3x+7\,\text{m s}^{-1}$.

(a) Find an expression for the acceleration.

(b) If the initial displacement is 1 m to the right of the origin, find the displacement as a function of time.

Solution

(a) $v=3x+7$

$$\frac{dv}{dx}=3$$

Use $\ddot{x}=v\frac{dv}{dx}$: $\quad \ddot{x}=(3x+7)\times 3$

$$=3(3x+7)\,\text{m s}^{-2}$$

(b) $\frac{dx}{dt}=3x+7$

Reciprocal of both sides: $\quad \frac{dt}{dx}=\frac{1}{3x+7}$

Integrate with respect to x: $\quad t=\int\frac{dx}{3x+7}$

$$t=\frac{1}{3}\log_e(3x+7)+C$$

At $t=0$, $x=1$: $\quad C=-\frac{1}{3}\log_e 10$

$$\therefore \quad t=\frac{1}{3}\log_e(3x+7)-\frac{1}{3}\log_e 10$$

$$t=\frac{1}{3}\log_e\left(\frac{3x+7}{10}\right)$$

$$3t=\log_e\left(\frac{3x+7}{10}\right)$$

$$\left(\frac{3x+7}{10}\right)=e^{3t}$$

$$3x+7=10e^{3t}$$

$$x=\frac{1}{3}\left(10e^{3t}-7\right)$$

Example 2

A particle moves in a straight line so that at time t its displacement from a fixed origin is x and its velocity is v. If $\ddot{x} = 3x^2$ and $v = -\sqrt{2}$, $x = 1$ when $t = 0$, find x as a function of t.

Solution

$\ddot{x} = 3x^2$ gives acceleration as a function of x, so to find v as a function of x:

Use $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$: $\quad \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 3x^2$

Integrate with respect to x: $\quad \frac{1}{2}v^2 = \int 3x^2\,dx$

$$\frac{1}{2}v^2 = x^3 + C_1$$

At $x = 1$, $v = -\sqrt{2}$: $\quad 1 = 1 + C_1$

$$C_1 = 0$$

$\therefore \quad \frac{1}{2}v^2 = x^3$

$$v^2 = 2x^3$$

$$v = \pm\sqrt{2x^3}$$

The conditions include $x = 1$, $v = -\sqrt{2}$, which means that v is negative when $x = 1$.
To satisfy the initial conditions you must take the negative square root.

$\therefore \quad v = -\sqrt{2}x^{\frac{3}{2}}$

Now $v = \frac{dx}{dt}$: $\quad \frac{dx}{dt} = -\sqrt{2}x^{\frac{3}{2}}$

Reciprocal of both sides: $\quad \frac{dt}{dx} = -\frac{1}{\sqrt{2}}x^{-\frac{3}{2}}$

Integrate with respect to t: $\quad t = -\frac{1}{\sqrt{2}}\int x^{-\frac{3}{2}}\,dx$

$$t = -\frac{1}{\sqrt{2}} \times \left(-\frac{2}{1}\right)x^{-\frac{1}{2}} + C_2$$

$$t = \sqrt{2}x^{-\frac{1}{2}} + C_2$$

At $x = 1$, $t = 0$: $\quad 0 = \sqrt{2} + C_2$

$$C_2 = -\sqrt{2}$$

$\therefore \quad t = \sqrt{2}x^{-\frac{1}{2}} - \sqrt{2}$

Find x in terms of t: $\quad \sqrt{2}x^{-\frac{1}{2}} = t + \sqrt{2}$

$$x^{-\frac{1}{2}} = \frac{t + \sqrt{2}}{\sqrt{2}}$$

$$x^{\frac{1}{2}} = \frac{\sqrt{2}}{t + \sqrt{2}}$$

$$x = \frac{2}{\left(t + \sqrt{2}\right)^2}$$

Example 4

A particle moves in a straight line so that at time t its displacement from a fixed origin is x and its velocity is v. If $\ddot{x} = -e^{-\frac{x}{2}}$ and $v = 2$, $x = 0$ when $t = 0$, find x as a function of t.

Solution

Use $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$: $\quad \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -e^{-\frac{x}{2}}$

Integrate with respect to x: $\quad \frac{1}{2}v^2 = -\int e^{-\frac{x}{2}}\,dx$

$$\frac{1}{2}v^2 = 2e^{-\frac{x}{2}} + C_1$$

At $x = 0$, $v = 2$: $\quad 2 = 2 + C_1$

$$C_1 = 0$$

$\therefore \quad \frac{1}{2}v^2 = 2e^{-\frac{x}{2}}$

$$v^2 = 4e^{-\frac{x}{2}}$$

$$v = \pm 2e^{-\frac{x}{4}}$$

The conditions include $x = 0$, $v = 2$, which means that v is positive when $x = 0$; also $e^{-\frac{x}{4}} > 0$ for all x. To satisfy the initial conditions you must take the positive square root.

$\therefore \quad v = 2e^{-\frac{x}{4}}$

Now $v = \frac{dx}{dt}$: $\quad \frac{dx}{dt} = 2e^{-\frac{x}{4}}$

Reciprocal of both sides: $\quad \frac{dt}{dx} = \frac{1}{2}e^{\frac{x}{4}}$

Integrate with respect to t: $\quad t = \frac{1}{2}\int e^{\frac{x}{4}}\,dx$

$$t = \frac{1}{2} \times 4e^{\frac{x}{4}} + C_2$$

$$t = 2e^{\frac{x}{4}} + C_2$$

At $t = 0$, $x = 0$: $\quad 0 = 2 + C_2$

$$C_2 = -2$$

$\therefore \quad t = 2e^{\frac{x}{4}} - 2$

Find x in terms of t: $\quad e^{\frac{x}{4}} = \frac{t+2}{2}$

$$\frac{x}{4} = \log_e\left(\frac{t+2}{2}\right)$$

$$x = 4\log_e\left(\frac{t+2}{2}\right)$$