AREA BETWEEN TWO CURVES

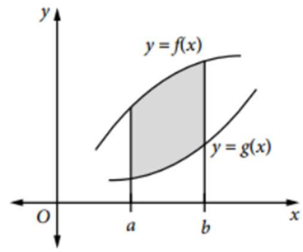
Area between two curves—general result

If f and g are two continuous functions whose graphs do not intersect in the interval $a \le x \le b$, and $f(x) > g(x)$ over this interval, then the area of the region bounded by the two curves and the ordinates $x = a$ and $x = b$ is given by:

$$\text{Area} = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$= \int_a^b \big(f(x) - g(x)\big)\,dx$$

This result also applies if the two curves intersect only at $x = a$ and $x = b$.

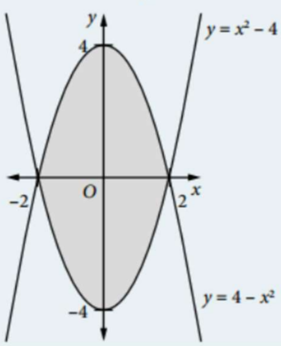


Example 15

Sketch the region bounded by the curves $y = 4 - x^2$ and $y = x^2 - 4$. By evaluating the appropriate definite integrals, calculate the area of this region.

Solution

Sketch the region:



Both curves cut the x-axis at $x = \pm 2$.

Call the area above the x-axis A_1 and the area below the x-axis A_2.

A_1 is bounded by $y = 4 - x^2$ and the x-axis, hence:

$$A_1 = \int_{-2}^{2} \left(4 - x^2\right) dx$$

You can see from the sketch that this area is positive, so the absolute value bars can be left out.

A_2 is bounded by $y = x^2 - 4$ and the x-axis, hence:

$$A_2 = \left| \int_{-2}^{2} \left(x^2 - 4\right) dx \right|$$

You can see from the sketch that the integral for this area will be negative, so you must include absolute value bars.

Evaluate the integrals:

$$\begin{aligned} A_1 &= \int_{-2}^{2} \left(4 - x^2\right) dx \\ &= \left[4x - \frac{x^3}{3}\right]_{-2}^{2} \\ &= \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right) \\ &= \frac{32}{3} \\ &= 10\frac{2}{3} \end{aligned}$$

$$\begin{aligned} A_2 &= \left| \int_{-2}^{2} \left(x^2 - 4\right) dx \right| \\ &= \left| \left[\frac{x^3}{3} - 4x\right]_{-2}^{2} \right| \\ &= \left| \left(\frac{8}{3} - 8\right) - \left(-\frac{8}{3} + 8\right) \right| \\ &= \left| -\frac{32}{3} \right| \\ &= 10\frac{2}{3} \end{aligned}$$

$$\begin{aligned} \text{Area of shaded region} &= A_1 + A_2 \\ &= 10\frac{2}{3} + 10\frac{2}{3} = 21\frac{1}{3} \text{ units}^2 \end{aligned}$$

Example 16

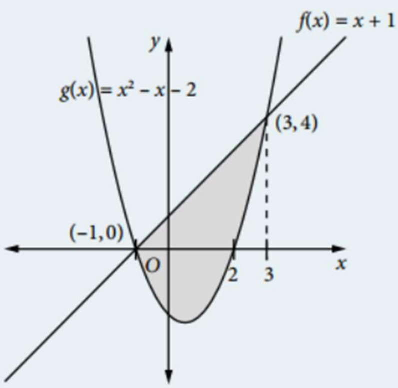
Calculate the area of the region enclosed by the graphs of $f(x) = x + 1$ and $g(x) = x^2 - x - 2$.

Solution

Find the x-values of the points of intersection of the two curves by solving $f(x)$ and $g(x)$ simultaneously:

$$x^2 - x - 2 = x + 1$$
$$x^2 - 2x - 3 = 0$$
$$(x+1)(x-3) = 0$$
$$x = -1, 3$$
$$y = 0, 4$$

Sketch the region:



$f(x) \geq g(x)$ for $-1 \leq x \leq 3$. Area of the region A is given by:

$$A = \int_{-1}^{3} f(x)\,dx - \int_{-1}^{3} g(x)\,dx$$

or $$A = \int_{-1}^{3} \big(f(x) - g(x)\big)\,dx$$

Hence: $$A = \int_{-1}^{3} \Big((x+1) - \big(x^2 - x - 2\big)\Big)\,dx$$

$$= \int_{-1}^{3} \left(-x^2 + 2x + 3\right)dx$$

$$= \left[-\frac{x^3}{3} + x^2 + 3x\right]_{-1}^{3}$$

$$= (-9 + 9 + 9) - \left(\frac{1}{3} + 1 - 3\right) = 10\frac{2}{3}$$

The area between the curves is $10\frac{2}{3}$ units^2.