RATES OF CHANGE AND THEIR APPLICATION - CHAPTER REVIEW

1 A particle is moving along the x-axis and is initially at the origin. Its velocity v metres per second at time t seconds is given by $v = \frac{2t}{9+t^2}$.

(a) What is the initial velocity of the particle?

(b) Find an expression for the acceleration of the particle.

(c) When is the acceleration zero?

(d) What is the maximum velocity attained by the particle and when does it occur?

a) At $t=0$ $\quad v(0) = \frac{2\times 0}{9+0} = 0$

b) $a(t) = \frac{dv}{dt} = \frac{2(9+t^2) - 2t\times 2t}{(9+t^2)^2} = \frac{18-2t^2}{(9+t^2)^2} = \frac{2(9-t^2)}{(9+t^2)^2}$

c) $a(t) = 0$ when $9 - t^2 = 0$, i.e $t = 3$ (negative solution is impossible)

d) at $t=3$ $\quad v(3) = \frac{2\times 3}{9+3^2} = \frac{6}{18} = \frac{1}{3}$ m s^{-1}

2 The growth of the number of internet users in the USA was modelled as an exponential function $N = Ae^{kt}$, where N is the estimate for the number of internet users (in millions) and t is the time in years after 1 January 2001.

(a) At the start of 2001 ($t = 0$) there were 124 million internet users in the USA. At the start of 2009 there were 220 million. Find A and k.

(b) How many internet users would you expect there to be at the start of 2012?

(c) In what year would you expect the number of internet users in the USA to first exceed 300 million?

a) $124 = Ae^0 = A$ so $A = 124$ $\quad$ At $t=8$ $\quad N = 220 = 124e^{k8}$

so $8k = \ln\left(\frac{220}{124}\right)$

$k = \frac{1}{8}\ln\left(55/31\right)$

b) At $t=11$ $\quad N = 124e^{\frac{1}{8}\ln(55/31)\times 11} = 124\left(\frac{55}{31}\right)^{11/8} \simeq 273$

c) $N = 124e^{\frac{1}{8}\ln(55/31)t} = 124\left(\frac{55}{31}\right)^{t/8}$ so $300 = 124\left(\frac{55}{31}\right)^{t/8}$

$\Leftrightarrow \left(\frac{55}{31}\right)^{t/8} = \frac{300}{124} = \frac{75}{31} \Rightarrow \ln\left(\frac{55}{31}\right)^{t/8} = \ln\frac{75}{31}$

so $\frac{t}{8} = \frac{\ln 75 - \ln 31}{\ln 55 - \ln 31}$ $\quad$ so $t = 8\left[\frac{\ln 75 - \ln 31}{\ln 55 - \ln 31}\right] \simeq 12.3$ so about 2013

3 A full water tank holds 4000 litres. When the tap is turned on, water flows out from the tank at a rate of $\frac{dV}{dt} = 110 + 17t - t^2$ litres per minute, where V is the volume in litres and t is the time in minutes since the tap was turned on.

(a) At what time is the tank emptying at a rate of 50 litres per minute?
(b) At what time does the water stop flowing out of the tank?

a) $\frac{dV}{dt} = 50$ when $110 + 17t - t^2 = 50 \Leftrightarrow -t^2 + 17t + 60 = 0$

$\Delta = 17^2 - 4 \times (-1) \times 60 = 529 = 23^2 \quad t = \frac{-17-23}{-2} = 20_{min}$

b) $\frac{dV}{dt} = 0$ when $110 + 17t - t^2 = 0$

$\Delta = 17^2 - 4 \times (-1) \times 110 = 729 = 27^2$

$t = \frac{-17-27}{-2} = 22$ min.

5 The number N of bacteria in a colony grows according to the rule $\frac{dN}{dt} = kN$. If the original number increases from 5000 to 10 000 in 6 days, find the number of bacteria after another 6 days.

$N = N_0 e^{kt}$ At $t = 0$ $N = 5000$ so $N_0 = 5000$

At $t = 6$, $N = 10{,}000$, so $10000 = 5000\, e^{k \times 6}$

$\Rightarrow e^{6k} = 2 \Rightarrow \ln(e^{6k}) = \ln 2 \Rightarrow 6k = \ln 2$ so $k = \frac{1}{6}\ln 2$

So $N = 5{,}000\, e^{\frac{1}{6}\ln 2 \times t} = 5{,}000 \times 2^{t/6}$

At $t = 12$ $N = 5{,}000 \times 2^{12/6} = 5{,}000 \times 2^2 = 20{,}000$

6 A population of size N is decreasing according to the rule $\frac{dN}{dt} = -\frac{N}{40}$, where t is the time in days. If the population is initially of size N_0, find how much time it takes for the size to be halved (rounded to the next day).

$N = N_0 e^{-kt}$ $\quad \frac{dN}{dt} = N_0 \times (-k) e^{-kt} = -kN$ so $k = \frac{1}{40}$

$N = N_0 e^{-t/40}$

for $N = \frac{N_0}{2}$, $\frac{N_0}{2} = N_0 e^{-t/40} \Rightarrow e^{-t/40} = \frac{1}{2}$

$\Rightarrow \ln\left[e^{-t/40}\right] = \ln\left(\frac{1}{2}\right) \Rightarrow -\frac{t}{40} = -\ln 2$

so $t = 40 \ln 2 \simeq 28$ days

7 A radioactive substance decays at a rate that is proportional to the mass of radioactive substance present at any time. If 10% decays in 400 years, what percentage of the original radioactive mass will remain after 1000 years?

$\frac{dN}{dt} = -kN$ so $N = N_0 e^{-kt}$

At $t = 400$, $N = 0.9 \times N_0$ so $0.9 N_0 = N_0 e^{-k \times 400}$

so $e^{-400k} = 0.9 \Rightarrow -400k = \ln 0.9 = \ln \frac{9}{10} = -\ln\left(\frac{10}{9}\right)$

so $k = \frac{1}{400} \ln\left(\frac{10}{9}\right)$ so $N = N_0 e^{\frac{1}{400}\ln\left(\frac{9}{10}\right) t} = N_0 \left(\frac{9}{10}\right)^{t/400}$

At $t = 1000$, $N = N_0 \left(\frac{9}{10}\right)^{1000/400} = N_0 \left(\frac{9}{10}\right)^{5/2}$

so $\frac{N}{N_0} = \left(\frac{9}{10}\right)^{5/2} \simeq 77\%$ approx left

8 The radius of a cylinder increases at a constant rate of 0.1 cm per minute while its height remains constant at 10 cm. At what rate is the volume of the cylinder increasing when the radius is 2 cm?

$r = kt + r_0 = 0.1t + r_0$. $\quad \frac{dr}{dt} = 0.1$

$V = \pi r^2 \times h \qquad \frac{dV}{dt} = \pi h \times \frac{dr^2}{dt} = \pi h\, 2r \frac{dr}{dt} = \pi h 2r \times 0.1$

So $\frac{dV}{dt} = \pi \times \cancel{10} \times 2 \times 2 \times 0.\cancel{1} = 4\pi$ cm³/min.

9 Rain is falling and collects in an inverted cone so that the volume collected increases at a constant rate of 4π cm³ per hour. If the radius r of the cone is half its height h, find the rate (in cm per hour) at which the height is increasing when $h = 3$.

$\frac{dV}{dt} = 4\pi$ cm³/hr

$V = \frac{1}{3} \times \pi r^2 \times h = \frac{1}{3} \times \pi \left(\frac{h}{2}\right)^2 \times h = \frac{\pi h^3}{12}$

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{1}{\frac{dV}{dh}} \times 4\pi = \frac{1}{\frac{3\pi h^2}{12}} \times 4\pi = \frac{48}{3h^2}$

so $\frac{dh}{dt} = \frac{16}{h^2}$

At $h=3$, $\frac{dh}{dt} = \frac{16}{9}$ cm/hr

10 A cup of hot coffee at a temperature T°C loses heat in a cooler environment. It cools according to the law $\frac{dT}{dt} = -k(T - T_0)$, where t is the time elapsed in minutes, T_0 is the temperature of the environment in degrees Celsius and k is a constant.

(a) At recess, Mr Masters makes his cup of coffee with water at 100°C. The temperature in the staff room is 25°C when he places his coffee on his desk. Three minutes later his coffee is just the temperature he likes it, 75°C. Find the value of k.

(b) Before Mr Masters gets a chance to drink his coffee, he leaves the staff room to help a student with a maths problem about exponential growth and decay. What is the temperature of the coffee when he returns, 5 minutes later?

a) $T = T_0 + Ae^{-kt} = 25 + Ae^{-kt}$

At $t=0$ $T = 100$, so $100 = 25 + A$ so $A = 75$.

So $T = 25 + 75e^{-kt}$

At $t = 3$ $T = 75$, so $75 = 25 + 75e^{-k\times 3}$

so $e^{-3k} = \frac{50}{75} = \frac{2}{3}$ so $-3k = \ln\left(\frac{2}{3}\right)$

so $k = \frac{1}{3}\ln\left(\frac{3}{2}\right) \simeq 0.135$ so $T = 25 + 75e^{-\frac{1}{3}\ln(3/2)t}$

$T = 25 + 75\left(\frac{3}{2}\right)^{-t/3}$

b) At $t = 8$ $T = 25 + 75\left(\frac{3}{2}\right)^{-8/3} \simeq 50.4°C$

13 If $P = \frac{100}{V}$ and $\frac{dV}{dt} = 4$, find the expression for $\frac{dP}{dt}$.

$\frac{dP}{dt} = \frac{dP}{dV} \times \frac{dV}{dt}$ (Chain rule) so $\frac{dP}{dt} = 4 \times \frac{dP}{dV}$

$P = \frac{100}{V}$ so $\frac{dP}{dV} = -\frac{100}{V^2}$ so $\frac{dP}{dt} = 4 \times \left(-\frac{100}{V^2}\right)$

$\frac{dP}{dt} = -\frac{400}{V^2}$

14 **(a)** If $x = 2\left(t + \frac{1}{t}\right)$, $y = 2\left(t - \frac{1}{t}\right)$, find an expression for $\frac{dy}{dx}$ in terms of t.

(b) Find $\frac{d^2y}{dx^2}$ as a function of t.

a) $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ (Chain rule) so $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{1}{\frac{dx}{dt}}$

$x = 2t + \frac{2}{t}$ so $\frac{dx}{dt} = 2 - \frac{2}{t^2}$ so $\frac{dy}{dx} = \left[2 + \frac{2}{t^2}\right] \times \frac{1}{\left[2 - \frac{2}{t^2}\right]}$

$y = 2t - \frac{2}{t}$ so $\frac{dy}{dt} = 2 + \frac{2}{t^2}$

$\frac{dy}{dx} = \frac{1 + \frac{1}{t^2}}{1 - \frac{1}{t^2}} = \frac{t^2+1}{t^2-1}$

b) $\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dt}\left[\frac{dy}{dx}\right] \times \frac{dt}{dx} = \frac{d}{dt}\left[\frac{t^2+1}{t^2-1}\right] \times \frac{1}{\frac{dx}{dt}}$

so $\frac{d^2y}{dx^2} = \frac{2t(t^2-1) - 2t(t^2+1)}{(t^2-1)^2} \times \frac{1}{2 - \frac{2}{t^2}}$

quotient rule

$\frac{d^2y}{dx^2} = \frac{-4t}{(t^2-1)^2} \times \frac{1}{2 - \frac{2}{t^2}} = \frac{2t}{(t^2-1)^2\left(\frac{1}{t^2} - \cancel{1}\right)} = \frac{2t^3}{(t^2-1)^2(1-t^2)}$

$\frac{d^2y}{dx^2} = \frac{2t^3}{(1-t^2)^3}$

15 A spherical balloon is being filled with air at the rate of $100\,cm^3/min$. At what rate is the radius of the balloon increasing when the radius is 5 cm?

$\frac{dV}{dt} = 100\ cm^3/min$ $\quad V = \frac{4}{3}\pi r^3$

so $\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt} = \frac{1}{dV/dr} \times 100 = \frac{1}{\frac{4}{3}\pi \times 3r^2} \times 100$

so $\frac{dr}{dt} = \frac{100}{4\pi r^2}$

At $r = 5\ cm$, $\frac{dr}{dt} = \frac{100}{4\pi \times 5^2} = \frac{100}{\pi \times 4 \times 25} = \frac{1}{\pi}\ cm/min.$

17 A point P moves on the curve $y = x^3$ so that its x-coordinate increases at a constant rate of 5 units per second. When $x = 1$:

(a) at what rate is the y-coordinate of P increasing

(b) at what rate is the gradient of the curve increasing?

$\frac{dx}{dt} = 5$ units/s. $\quad$ a) $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} = 5\frac{dy}{dx}$

so $\frac{dy}{dt} = 5 \times \frac{d}{dx}(x^3) = 5 \times 3x^2$

when $x = 1$ $\quad \frac{dy}{dt} = 5 \times 3 \times 1^2 = 15$ units/second.

b) $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt} \times \frac{1}{\frac{dx}{dt}} = 15 \times \frac{1}{5} = 3$ units/s.

18 A vessel is shaped so that when the depth of water in it is x cm, the volume of the water is V cm^3, where $V = 108x + x^3$. Water is poured into the vessel at a constant rate of 30 cm^3/s. At what rate is the water level rising when its depth is 8 cm?

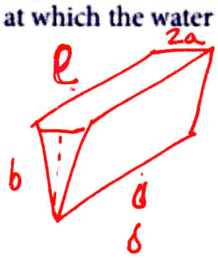
$\frac{dV}{dt} = 30$ cm^3/s. $\frac{dx}{dt} = \frac{dx}{dV} \times \frac{dV}{dt} = \frac{1}{\frac{dV}{dx}} \times 30$

$V = 108x + x^3$ so $\frac{dV}{dx} = 108 + 3x^2$

So $\frac{dx}{dt} = \frac{1}{108 + 3x^2} \times 30 = \frac{10}{36 + x^2}$

when $x = 8$, $\frac{dx}{dt} = \frac{10}{36 + 8^2} = \frac{10}{100} = \frac{1}{10}$ cm/s.

19 A trough l metres long has a cross-section in the shape of an isosceles triangle with base length $2a$ metres and height b metres. Water leaks from the trough at a constant rate of c m^3/min. Find the rate at which the water level is falling when the depth of the water is $\frac{b}{2}$ metres.



$V = l \times \frac{2a \times b}{2} = abl$

$\frac{dV}{dt} = c$ m^3/min.

$\frac{db}{dt} = \frac{db}{dV} \times \frac{dV}{dt} = c \times \frac{db}{dV} = c \times \frac{1}{\frac{dV}{db}}$

$V = abl$ so $\frac{dV}{db} = al$ so $\frac{db}{dt} = \frac{c}{al}$

The water level is falling at $\frac{c}{al}$ m/min.

20 A metal sphere is dissolving in acid. It remains spherical and the rate at which its volume decreases is proportional to its surface area. Show that the radius of the sphere decreases at a constant rate.

$$\frac{dV}{dt} = kS$$

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt} = kS \times \frac{dr}{dV} = kS \times \frac{1}{\frac{dV}{dr}}$$

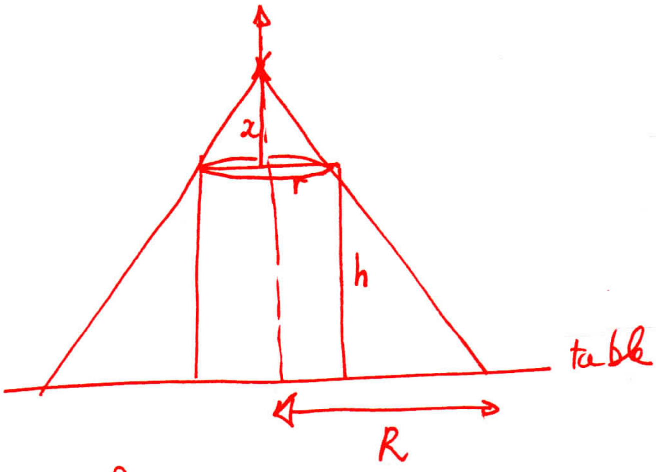
$$\frac{dr}{dt} = 4\pi r^2 \times \frac{k}{\frac{dV}{dr}}$$

$V = \frac{4}{3}\pi r^3$, so $\frac{dV}{dr} = \frac{4}{3}\pi \times 3r^2 = 4\pi r^2$

so $\frac{dr}{dt} = 4\pi r^2 \times \frac{k}{4\pi r^2} = k.$

Therefore $\frac{dr}{dt}$ = constant, i.e. the radius of the sphere decreases at a constant rate.

21 A circular cylinder with height 6 cm and base radius 4 cm sits on a table with its axis vertical. A point source of light moves vertically up at a speed of 3 cm/s above the central axis of the cylinder, thus casting a circular shadow on the table. Find the rate at which the radius of the shadow is decreasing when the light is 4 cm above the top of the cylinder.



we look for $\dfrac{dR}{dt}$

we know $\dfrac{dx}{dt}$.

$$\frac{dR}{dt} = \frac{dR}{dx} \times \frac{dx}{dt} = 3 \times \frac{dR}{dx}$$

By similarity of the triangles, $\dfrac{x}{r} = \dfrac{h+x}{R} \Leftrightarrow \dfrac{R}{R+x} = \dfrac{r}{x}$

or $R = r \times \dfrac{h+x}{x} = r\left(\dfrac{h}{x}+1\right) = \dfrac{rh}{x} + r$

so $\dfrac{dR}{dx} = -\dfrac{hr}{x^2}$

So $\dfrac{dR}{dt} = -\dfrac{3hr}{x^2}$

When $x = 4$, $\dfrac{dR}{dt} = -\dfrac{3 \times 6 \times 4}{4^2} = -\dfrac{18}{4} = -\dfrac{9}{2} = -4.5$

$\dfrac{dR}{dt} = -4.5$ cm/s.