USING INDUCTION TO PROVE FIRST-ORDER RECURSIVE FORMULAE

1 If $u_{n+1} = 3u_n + 4$ and $u_1 = 1$, use mathematical induction to prove that $u_n = 3^n - 2$ for all positive integers n.

Step 1 For $n = 1$ $u_1 = 1$ and $3^1 - 2 = 3 - 2 = 1$ too
so it's true for $n = 1$

Step 2 Assume it's true for $n = k$, i.e. $u_k = 3^k - 2$

Then $u_{n+1} = 3u_n + 4$

$= 3[3^k - 2] + 4$

$= 3^{k+1} - 6 + 4$

$= 3^{k+1} - 2$

So it's also true for $(k+1)$ then

Step 3. it's true for $n = 1$ as shown in Step 1
it's true for $(k+1)$ if it's true for k as shown in Step 2

$\therefore$ by induction, it's true for all positive integers n.

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4 A sequence is defined recursively as $u_1 = 5$, $u_2 = 13$, $u_n = 5u_{n-1} - 6u_{n-2}$ for $n \geq 3$. By induction, prove that $u_n = 2^n + 3^n$ for all positive integers n.

Step 1 $u_1 = 5$ whereas $2^1 + 3^1 = 5$ too

So it's true for $n = 1$ / $n=2$ $u_2 = 13$ whereas $2^2+3^2 = 13$ too
so true for $n=2$

Step 2 Assume it's true for $n = k$ and $n = k-1$

Then $u_{k+1} = 5u_k - 6u_{k-1}$

$$u_{k+1} = 5(2^k + 3^k) - 6(2^{k-1} + 3^{k-1})$$

$$= 2^{k-1}(5 \times 2 - 6) + 3^{k-1}(5 \times 3 - 6)$$

$$= 2^{k-1} \times 4 + 3^{k-1} \times 9$$

$$= 2^{k-1} \times 2^2 + 3^{k-1} \times 3^2$$

$$= 2^{k+1} + 3^{k+1}$$

So it's true for $(k+1)$ if it's true for k and for $(k-1)$

Step 3 it's true for $n = 1$ and for $n = 2$

it's true for $(k+1)$ if it's true for k and for $(k-1)$

$\therefore$ by induction, it's true for all positive n. integers

7 The Fibonacci sequence is defined as $u_1 = 1$, $u_2 = 1$, $u_{n+2} = u_{n+1} + u_n$ for all positive integers $n \geq 1$.

Prove by induction that $u_n < \left(\frac{5}{3}\right)^n$ for all positive integers n.

Step 1 $n=1$ $u_1 = 1$ whereas $\left(\frac{5}{3}\right)^1 = \frac{5}{3} > 1$ indeed.

so true for $n=1$

$n=2$ $u_2 = 1$ whereas $\left(\frac{5}{3}\right)^2 = \frac{25}{9} > 1$ indeed

so true for $n=2$

Step 2 we assume it's true for $n=k$ and for $n=k+1$ too.

In that case $u_{k+2} = u_{k+1} + u_k$ But $u_{k+1} < \left(\frac{5}{3}\right)^{k+1}$

$u_k < \left(\frac{5}{3}\right)^k$

So $u_{k+2} < \left(\frac{5}{3}\right)^{k+1} + \left(\frac{5}{3}\right)^k$

$$u_{k+2} < \left(\frac{5}{3}\right)^{k+1}\left[\left(\frac{5}{3}\right)^{-1} + 1\right]$$

$$u_{k+2} < \left(\frac{5}{3}\right)^{k+1}\left(1 + \frac{3}{5}\right)$$

$$u_{k+2} < \left(\frac{5}{3}\right)^{k+1}\left(\frac{8}{5}\right)$$

But $\frac{8}{5} = 1.6$

and $\frac{5}{3} = 1.\dot{6}$

$\therefore \frac{8}{5} < \frac{5}{3}$

$$\therefore u_{k+2} < \left(\frac{5}{3}\right)^{k+1}\left(\frac{8}{5}\right) < \left(\frac{5}{3}\right)^{k+1} \times \left(\frac{5}{3}\right)$$

$$\therefore u_{k+2} < \left(\frac{5}{3}\right)^{k+2}$$

Step 3 : it's true for $n=1$ and $n=2$

it's true for $(k+2)$ if it's true for k and $k+1$

$\therefore$ by induction, it's true for all positive integers n.

8 If $a_0 = 1$, $a_1 = 6$ and $a_n = 6a_{n-1} - 9a_{n-2}$, use mathematical induction to prove that $a_n = 3^n + n3^n$.

Step 1 $n = 0$ $a_0 = 1$ whereas $3^0 + 0 \times 3^0 = 1$ true for $n = 0$

$n = 1$ $a_1 = 6$ whereas $3^1 + 1 \times 3^1 = 3 + 3 = 6$ true for $n = 1$

Step 2 We assume it's true for $(k-2)$ and $(k-1)$

then $a_k = 6a_{k-1} - 9a_{k-2}$

$$a_k = 6\left[3^{k-1} + (k-1)3^{k-1}\right] - 9\left[3^{k-2} + (k-2)3^{k-2}\right]$$

$$a_k = 3^{k-2}\left[6 \times 3 - 9\right] + 3^{k-2}\left[6(k-1) \times 3 - 9(k-2)\right]$$

$$a_k = 9 \times 3^{k-2} + 3^{k-2}\left[9k - 18 + 18\right]$$

$$a_k = 3^k + 3^{k+2} \times 3^2\left[k\right]$$

$$a_k = 3^k + k3^k$$

$\therefore$ it's true for k if it's true for $(k-2)$ and $(k-1)$

Step 3 . it's true for $n = 0$ and $n = 1$

it's true for k if it's true for $(k-2)$ and $(k-1)$

$\therefore$ by induction, it must be true for any positive integer.

9 **(a)** Prove by contradiction that $(4k+3)\sqrt{k} \le (4k+1)\sqrt{k+1}$ for all $k \ge 0$.

(b) Prove by induction that $\sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots+\sqrt{n} \le \frac{(4n+3)\sqrt{n}}{6}$ for all integers $n \ge 1$.

a) Assume that $(4k+3)\sqrt{k} > (4k+1)\sqrt{k+1}$ for all $k>0$

Taking the squares on both sides, we obtain:

$$(4k+3)^2 k > (4k+1)^2 (k+1)$$

$$\Leftrightarrow [16k^2+24k+9]k > [16k^2+8k+1](k+1)$$

$$\Leftrightarrow \cancel{16k^3}+\cancel{24k^2}+9k > \cancel{16k^3}+\cancel{16k^2}+\cancel{8k^2}+8k+k+1$$

$$\Leftrightarrow \quad 0 > 1$$ which is not true.

$\therefore$ the assumption was incorrect, we must have:

$$(4k+3)\sqrt{k} \leqslant (4k+1)\sqrt{k+1}$$

b) Step 1 $n=1$ $\quad \sqrt{1}=1$ whereas $\frac{(4\times1+3)\sqrt{1}}{6}=\frac{7}{6}$

so indeed $1<\frac{7}{6}$, it's true for $n=1$

Step 2 Assume it's true for $n=k$

In that case: $\sqrt{1}+\sqrt{2}+\ldots+\sqrt{k}+\sqrt{k+1} \leqslant \frac{(4k+3)\sqrt{k}}{6}+\sqrt{k+1}$

or $6\left[\sqrt{1}+\sqrt{2}+\ldots+\sqrt{k}+\sqrt{k+1}\right] \leqslant (4k+3)\sqrt{k}+6\sqrt{k+1}$

then $\_\_\_\_\_\_ \leqslant 6\sqrt{k+1}+(4k+1)\sqrt{k+1}$

$\_\_\_\_\_\_ \leqslant \sqrt{k+1}\,[4k+7]$

$\_\_\_\_\_\_ \leqslant \sqrt{k+1}\,[4(k+1)+3]$

So $\sqrt{1}+\sqrt{2}+\ldots+\sqrt{k}+\sqrt{k+1} \leqslant \frac{[4(k+1)+3]\sqrt{k+1}}{6}$

So it's true for $(k+1)$ if it's true for k

Step 3. it's true for $n=1$

it's true for $(k+1)$ if it's true for $n=k$

$\therefore$ by induction, it's true for all integers $n \geqslant 1$

10 If $4 = \frac{3}{u_1} = u_1 + \frac{3}{u_2} = u_2 + \frac{3}{u_3} = u_3 + \frac{3}{u_4} = \ldots$, prove by induction that $u_n = \frac{3^{n+1} - 3}{3^{n+1} - 1}$ for all positive integers n.

Step 1 $n = 1$ $u_1 = \frac{3}{4}$ whereas $\frac{3^{1+1} - 3}{3^{1+1} - 1} = \frac{6}{8} = \frac{3}{4}$

so true for $n = 1$

Step 2

The recurrence relation is $u_n + \frac{3}{u_{n+1}} = 4$

or $\frac{3}{u_{n+1}} = 4 - u_n$ or $\frac{u_{n+1}}{3} = \frac{1}{4 - u_n}$

so $\boxed{u_{n+1} = \frac{1}{4 - u_n}}$

Assume that $u_n = \frac{3^{n+1} - 3}{3^{n+1} - 1}$

In that case: $u_{n+1} = \frac{3}{4 - u_n} = \frac{3}{4 - \left(\frac{3^{n+1} - 3}{3^{n+1} - 1}\right)}$

so $u_{n+1} = \frac{3}{4 - \left(\frac{3^{n+1} - 3}{3^{n+1} - 1}\right)} \times \left(\frac{3^{n+1} - 1}{3^{n+1} - 1}\right)$

$u_{n+1} = \frac{3(3^{n+1} - 1)}{4(3^{n+1} - 1) - (3^{n+1} - 3)} = \frac{3(3^{n+1} - 1)}{3^{n+1}(4 - 1) - 4 + 3}$

so $u_{n+1} = \frac{3^{n+2} - 3}{3^{n+2} - 1}$ ∴ it's true for $(n+1)$ if it's true for n.

Step 3: * it's true for $n = 1$

* it's true for $(n+1)$ if it's true for n

∴ by induction, it's true $\forall n > 0$.