# APPROXIMATE METHODS OF INTEGRATION - TRAPEZOIDAL RULE

For many functions, you cannot easily find the primitive. In those cases, you can use numerical methods of integration to find the approximate value of a definite integral. There are many different methods of numerical approximation; computer software that evaluates definite integrals usually use sophisticated numerical methods.

## Approximating areas under curve

### Example 32

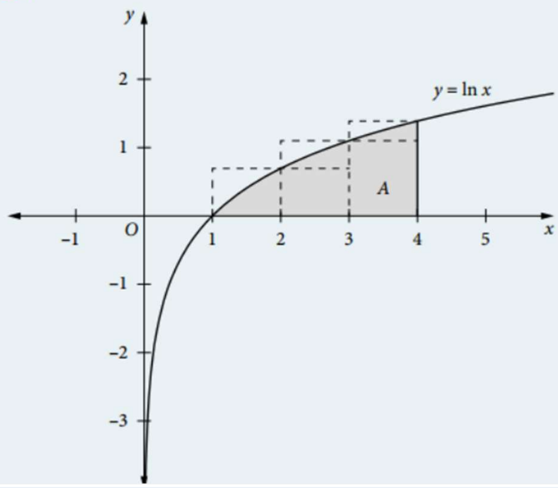
**(a)** Sketch the graph of $y = \ln x$ for $0 < x \le 5$.

**(b)** By drawing inner and outer rectangles and averaging the result, find an approximation for $\int_1^4 \ln x\,dx$, using three subintervals.

**(c)** Find an approximation for $\int_1^4 \ln x\,dx$ using trapezia and three subintervals.

### Solution

**(a)**




**(b)** For $1 \le x \le 4$ and three subintervals:

Area of inner rectangles

$= 1 \times 0 + 1 \times \ln 2 + 1 \times \ln 3 = \ln 6$

Area of outer rectangles

$= 1 \times \ln 2 + 1 \times \ln 3 + 1 \times \ln 4 = \ln 24$

Hence $\ln 6 < A < \ln 24$

Thus $\int_1^4 \ln x\,dx \approx \frac{\ln 6 + \ln 24}{2} = \frac{\ln 144}{2} = \ln 12$

**(c)** One side of each trapezium is very close to the graph of $y = \ln x$.

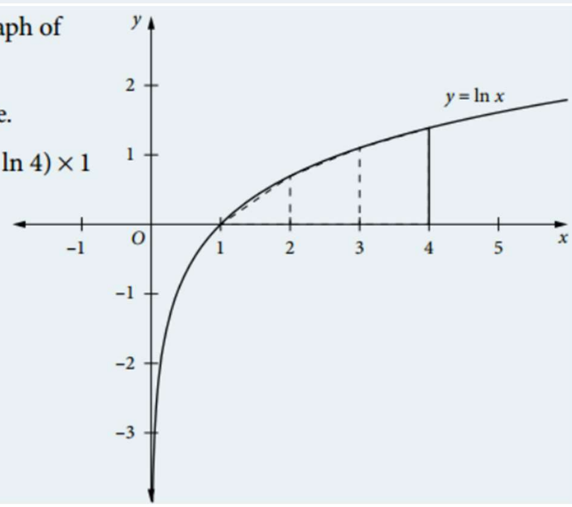
The first trapezium is really a right-angled triangle.

$$\int_1^4 \ln x\,dx \approx \frac{1}{2}\ln 2 + \frac{1}{2}(\ln 2 + \ln 3) \times 1 + \frac{1}{2}(\ln 3 + \ln 4) \times 1$$

$$= \frac{1}{2}(2\ln 2 + 2\ln 3 + \ln 4)$$

$$= 2\ln 2 + \ln 3$$

$$= \ln 12$$




The **trapezoidal rule** is a method where trapezia are used to approximate the area under the curve.

In Example 1 the area under the curve $y = x^2$ was approximated using rectangles drawn above and below the curve. The more rectangles used, the better the approximation obtained for the area. In Example 33 you will redo this problem using trapezia.

Recall the area of a trapezium: Area $= \frac{\text{sum of lengths of parallel sides}}{2} \times$ distance between them

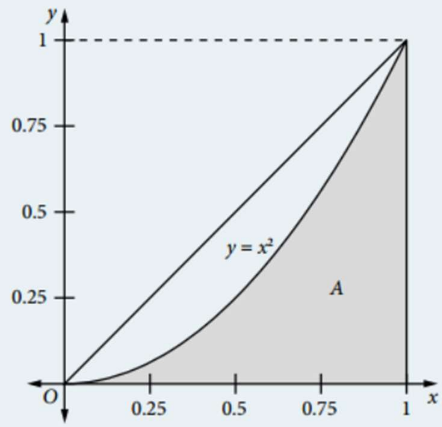
## Example 33

Calculate the area of the region bounded by the curve $y = x^2$, the $x$-axis and the ordinates at $x = 0$ and $x = 1$, using trapezia with:

**(a)** one subinterval **(b)** two subintervals **(c)** four subintervals.

## Solution

**(a)** Draw a diagram showing the region.

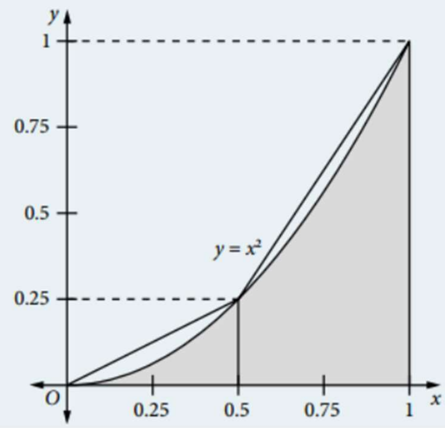



For $0 \le x \le 1$ and one subinterval, the trapezium is a triangle, but for consistency use the trapezium area formula.

$f(x) = x^2$: $f(0) = 0, f(1) = 1$, base $= 1 - 0 = 1$

$$\text{Area} = \frac{f(0) + f(1)}{2} \times (1 - 0)$$
$$= \frac{0 + 1}{2} \times 1 = 0.5 \text{ units}^2$$

**(b)** For $0 \le x \le 1$ and two subintervals:



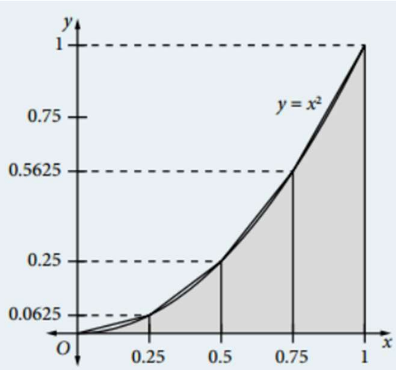

$f(x) = x^2$: $f(0) = 0, f(0.5) = 0.25, f(1) = 1$,
base $= 0.5 - 0 = 1 - 0.5 = 0.5$

$$\text{Area} = \frac{f(0) + f(0.5)}{2} \times 0.5 + \frac{f(0.5) + f(1)}{2} \times 0.5$$
$$= \frac{(0 + 0.25)}{2} \times 0.5 + \frac{(0.25 + 1)}{2} \times 0.5$$
$$= 0.0625 + 0.3125 = 0.375 \text{ units}^2$$

**(c)** For $0 \le x \le 1$ and four subintervals:

$f(x) = x^2$: $f(0) = 0, f(0.25) = 0.0625, f(0.5) = 0.25, f(0.75) = 0.5625, f(1) = 1$, base $= 0.25 - 0 = 0.25$

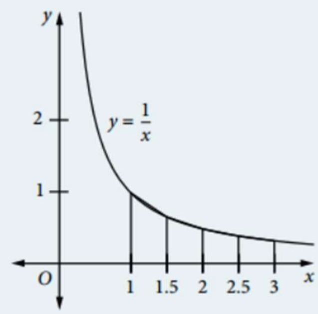
$$\text{Area} = \frac{f(0) + f(0.25)}{2} \times 0.25 + \frac{f(0.25) + f(0.5)}{2} \times 0.25 + \frac{f(0.5) + f(0.75)}{2} \times 0.25 + \frac{f(0.75) + f(1)}{2} \times 0.25$$
$$= \frac{0 + 0.0625}{2} \times 0.25 + \frac{0.0625 + 0.25}{2} \times 0.25 + \frac{0.25 + 0.5625}{2} \times 0.25 + \frac{0.5625 + 1}{2} \times 0.25$$
$$= 0.34375 \text{ units}^2$$




By evaluating $\int_0^1 x^2 \, dx$ you know that the exact area is $\frac{1}{3}$ units$^2$. You can see that increasing the number of subintervals (and hence trapezia) gives a better approximation to the area.

### Example 34

Use trapezia to find an approximation for the area under the curve $y = \frac{1}{x}$ between $x = 1$ and $x = 3$ using four subintervals.




### Solution

Sketch the region:

$f(x) = \frac{1}{x}$: $f(1) = 1$, $f(1.5) = \frac{2}{3}$, $f(2) = 0.5$, $f(2.5) = 0.4$, $f(3) = \frac{1}{3}$,

base $= 1.5 - 1 = 0.5$

$$\text{Area} = \frac{f(1)+f(1.5)}{2}\times 0.5 + \frac{f(1.5)+f(2)}{2}\times 0.5 + \frac{f(2)+f(2.5)}{2}\times 0.5 + \frac{f(2.5)+f(3)}{2}\times 0.5$$

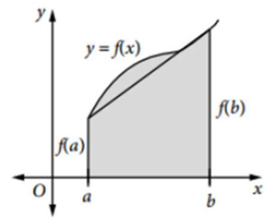
$$= \left(\left(1+\frac{2}{3}\right)+\left(\frac{2}{3}+\frac{1}{2}\right)+\left(\frac{1}{2}+\frac{2}{5}\right)+\left(\frac{2}{5}+\frac{1}{3}\right)\right)\times\frac{0.5}{2}$$

$$= \frac{1}{4}\left(1+2\times\frac{2}{3}+2\times\frac{1}{2}+2\times\frac{2}{5}+\frac{1}{3}\right)$$

$$= 1.11\dot{6} \text{ units}^2$$

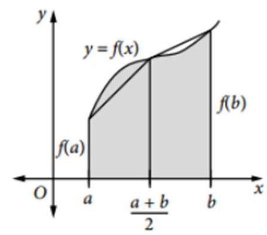
In this example you have found an approximation for the definite integral, $\int_1^3 \frac{dx}{x}$. The exact value of this integral is ln 3 so using trapezia gives a good approximation for this area.

## The trapezoidal rule

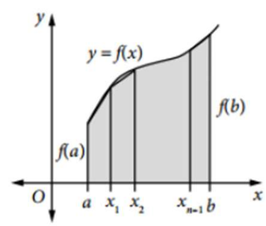



If $f(x)$ is a continuous function and $f(x) \geq 0$ on the interval $a \leq x \leq b$, then you can find an approximation to the definite integral $\int_a^b f(x)\,dx$ by dividing the area into a number of trapezia of equal width, where the parallel sides of the trapezia are ordinates.

- One trapezium: $$\int_a^b f(x)\,dx \approx (b-a)\left(\frac{f(a)+f(b)}{2}\right) = \frac{(b-a)}{2}\big(f(a)+f(b)\big)$$




- Two trapezia: $$\int_a^b f(x)\,dx \approx \frac{(b-a)}{2}\left(\frac{f(a)+f\left(\frac{a+b}{2}\right)}{2}\right)+\frac{(b-a)}{2}\left(\frac{f\left(\frac{a+b}{2}\right)+f(b)}{2}\right) = \frac{(b-a)}{4}\left(f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right)$$




- $n$ trapezia: the base of each trapezium is $\frac{b-a}{n} = h$. If $a = x_0$, $b = x_n$ and the points between are $x_1, x_2, \ldots x_{n-1}$, then the trapezoidal rule becomes: $$\int_a^b f(x)\,dx \approx \frac{(b-a)}{2n}\big(f(a)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-1})+f(b)\big)$$

**OR** $$\int_a^b f(x)\,dx \approx \frac{h}{2}\big(f(a)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-1})+f(b)\big)$$

where $h = \frac{b-a}{n}$.

For $h = \frac{b-a}{n}$, therefore:

$n = 1, h = b - a$: $$\int_a^b f(x)\,dx \approx \frac{(b-a)}{2}\big(f(a)+f(b)\big) = \frac{h}{2}\big(f(a)+f(b)\big)$$

$n = 2, h = \frac{b-a}{2}$: $$\int_a^b f(x)\,dx \approx \frac{h}{2}\left(f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right)$$

$n = n, h = \frac{b-a}{n}$: $$\int_a^b f(x)\,dx \approx \frac{h}{2}\big(f(a)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-1})+f(b)\big)$$ where $a = x_0$, $b = x_n$.

## Example 35

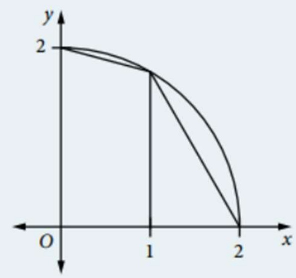
Evaluate $\int_0^2 \sqrt{4-x^2}\,dx$ using the trapezoidal rule with:

**(a)** two subintervals **(b)** four subintervals. **(c)** What is the exact value of this integral?

### Solution

**(a)** Use a table of values to sketch $f(x)=\sqrt{4-x^2}$ with two subintervals:

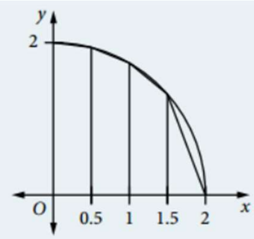
| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| $f(x)$ | 2 | $\sqrt{3}$ | 0 |




$$h=\frac{2-0}{2}=1:\quad \int_0^2 \sqrt{4-x^2}\,dx \approx \frac{1}{2}\left(f(0)+2f(1)+f(2)\right)$$
$$=\frac{1}{2}\left(2+2\sqrt{3}+0\right)$$
$$=2.732$$

**(b)** Use a table of values to sketch $f(x)=\sqrt{4-x^2}$ with four subintervals:

| $x$ | 0 | 0.5 | 1 | 1.5 | 2 |
|---|---|---|---|---|---|
| $f(x)$ | 2 | $\sqrt{3.75}$ | $\sqrt{3}$ | $\sqrt{1.75}$ | 0 |




$$h=\frac{2-0}{4}=0.5:\quad \int_0^2 \sqrt{4-x^2}\,dx \approx \frac{0.5}{2}\left(f(0)+2\left(f(0.5)+f(1)+f(1.5)\right)+f(2)\right)$$
$$=\frac{1}{4}\left(2+2\left(\sqrt{3.75}+\sqrt{3}+\sqrt{1.75}\right)+0\right)$$
$$=2.996$$

**(c)** The region is a quarter of a circle of radius 2, so the exact value of $\int_0^2 \sqrt{4-x^2}\,dx$ is $\pi$.

## Example 36

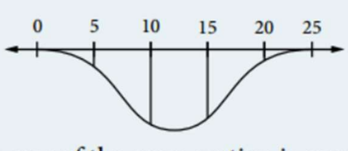
To measure the cross-sectional area of a river, a boat travels directly across the river and measures the river's depth as shown in the following table. The depth of the river at the banks is zero metres. Use the trapezoidal rule and all the values in the table to calculate the cross-sectional area of the river.

| Distance from riverbank (in metres) | 0 | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|---|
| Depth (in metres) | 0 | 1.2 | 5 | 4.8 | 1.3 | 0 |

### Solution

The distance from the bank is the independent variable and the depth of the river is the dependent variable.

$$\text{Area of cross-section} = \int_0^{25} f(x)dx \approx \frac{5}{2}\left(f(0)+2\left(f(5)+f(10)+f(15)+f(20)\right)+f(25)\right)$$
$$=\frac{5}{2}\left(0+2(1.2+5.0+4.8+1.3)+0\right)$$
$$=5\times 12.3$$
$$=61.5$$




The area of the cross-section is approximately $61.5\ \text{m}^2$.

If an average flow rate of the river was known, then the amount of water flowing past in a given time could be calculated.