RELATED RATES OF CHANGE

Problems with **related rates** arise when there is a function that relates two variables, e.g. x and y, where both variables are also functions of another variable, e.g. time t. For example, you may need to determine $\frac{dy}{dt}$ when $\frac{dx}{dt}$ is known. In such cases it is necessary to use the chain rule, $\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}$.

Remember that in a context like this, 'increasing' means a positive rate of change while 'decreasing' means a negative rate of change.

Example 13

(a) If $V=\frac{4}{3}\pi r^3$ and $\frac{dr}{dt}=5$, find the expression for $\frac{dV}{dt}$.

(b) If $S=4\pi r^2$ and $\frac{dr}{dt}=5$, find the expression for $\frac{dS}{dt}$.

(c) If $V=\frac{4}{3}\pi r^3$ and $S=4\pi r^2$, find the expression for $\frac{dV}{dS}$.

Solution

(a)
$$\frac{dV}{dr}=\frac{4}{3}\pi\times 3r^2$$
$$\frac{dV}{dr}=4\pi r^2$$
$$\frac{dV}{dt}=\frac{dV}{dr}\times\frac{dr}{dt}$$
$$\frac{dr}{dt}=5$$
$$\frac{dV}{dt}=4\pi r^2\times 5$$
$$\frac{dV}{dt}=20\pi r^2$$

(b)
$$\frac{dS}{dr}=8\pi r$$
$$\frac{dS}{dt}=\frac{dS}{dr}\times\frac{dr}{dt}$$
$$\frac{dr}{dt}=5$$
$$\frac{dS}{dt}=8\pi r\times 5$$
$$\frac{dS}{dt}=40\pi r$$

(c)
$$\frac{dV}{dr}=4\pi r^2,\ \frac{dS}{dr}=8\pi r$$
$$\frac{dV}{dS}=\frac{dV}{dr}\times\frac{dr}{dS}$$
$$\frac{dr}{dS}=\frac{1}{8\pi r}$$
$$\frac{dV}{dS}=4\pi r^2\times\frac{1}{8\pi r}$$
$$\frac{dV}{dS}=\frac{r}{2}$$

As $\frac{dr}{dt}$ is the same in parts **(a)** and **(b)**, these results can be used to find the answer to part **(c)** in this case.

Example 15

Given $x=t^2-1$ and $y=t^3$, find as functions of t: **(a)** $\frac{dy}{dx}$ **(b)** $\frac{d^2y}{dx^2}$

Solution

(a)
$$\frac{dx}{dt}=2t$$
$$y=t^3$$
$$\frac{dy}{dt}=3t^2$$
$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}$$
$$\frac{dy}{dx}=3t^2\times\frac{1}{2t}=\frac{3t}{2}$$

(b)
$$\frac{dy}{dx}=\frac{3t}{2},\ \frac{dx}{dt}=2t$$
$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\left(\frac{dy}{dx}\right)\times\frac{dt}{dx}$$
$$\frac{d}{dt}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\left(\frac{3t}{2}\right)=\frac{3}{2}$$
$$\frac{dt}{dx}=\frac{1}{2t}$$
$$\frac{d^2y}{dx^2}=\frac{3}{2}\times\frac{1}{2t}=\frac{3}{4t}$$

Example 16

A spherical balloon is being inflated so that its radius increases at the constant rate of 3 cm/min. At what rate is its volume increasing when the radius of the balloon is 5 cm?

Solution

If r is the radius of the balloon, its volume is $V=\frac{4}{3}\pi r^3$. Given $\frac{dr}{dt}=3$, need to find $\frac{dV}{dt}$ for $r=5$.

By the chain rule: $\frac{dV}{dt}=\frac{dV}{dr}\times\frac{dr}{dt}$

But: $V=\frac{4}{3}\pi r^3$

So: $\frac{dV}{dr}=4\pi r^2$

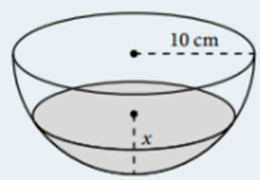
Thus: $\frac{dV}{dt}=\frac{dV}{dr}\times\frac{dr}{dt}=4\pi r^2\times 3$

For $r=5$: $\frac{dV}{dt}=4\pi\times 5^2\times 3$

$=300\pi\,\text{cm}^3/\text{min}$

Example 18

The volume of water in a hemispherical bowl of radius 10 cm is $V=\frac{1}{3}\pi x^2(30-x)$, where x cm is the depth of water at time t.
The bowl is being filled at a constant rate of $2\pi\,\text{cm}^3/\text{min}$.
At what rate is the depth increasing when the depth is 2 cm?



Solution

Given $\frac{dV}{dt}=2\pi$, need to find $\frac{dx}{dt}$ when $x=2$.
The related variables are V, x and t.

$V=\frac{1}{3}\pi x^2(30-x)$

$=10\pi x^2-\frac{1}{3}\pi x^3$

$\frac{dV}{dx}=20\pi x-\pi x^2=\pi x(20-x)$

Method 1

$\frac{dV}{dt}=\frac{dV}{dx}\times\frac{dx}{dt}$

$2\pi=\pi x(20-x)\times\frac{dx}{dt}$

$\frac{dx}{dt}=\frac{2}{x(20-x)}$

Method 2

$\frac{dx}{dt}=\frac{dx}{dV}\times\frac{dV}{dt}$

$\frac{dx}{dt}=\frac{1}{\pi x(20-x)}\times 2\pi$

$\frac{dx}{dt}=\frac{2}{x(20-x)}$

When $x=2$: $\frac{dx}{dt}=\frac{1}{18}$ cm/s

As an extension to this question, you might ask: 'At what depth is the depth increasing at a minimum rate?'

This is the same as asking: 'For what value of x is $\frac{dx}{dt}$ a minimum?'

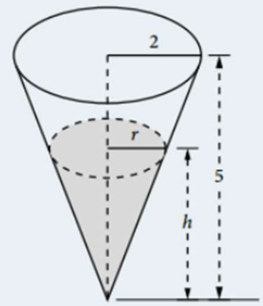
Note that the formula for the volume of water in the hemispherical bowl at any depth a can be calculated by finding the volume generated by rotating the circle with equation $x^2+y^2=100$ between $y=10-a$ and $y=10$.

RELATED RATES OF CHANGE

Example 17

A vessel containing water has the shape of an inverted right circular cone with base radius 2 m and height 5 m. The water flows out of the apex of the cone at a constant rate of 0.2 m^3/min. Find the rate at which the water level is dropping when the depth of the water is 4 m.

Solution

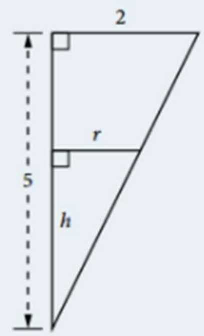


Let the depth of the water be h m, the radius of the cone at the water level be r m and the volume of the water be V m^3 at time t minutes.
The volume of the water at any time t is $V = \frac{1}{3}\pi r^2 h$.

Given $\frac{dV}{dt} = -0.2$, need to find $\frac{dh}{dt}$ when $h = 4$.

$\frac{dV}{dt}$ is negative: the volume is decreasing, because the water is flowing out of the vessel.

To find the link between r and h, use similar triangles.



From proportional sides: $\frac{r}{h} = \frac{2}{5}$

$$r = \frac{2h}{5}$$

Volume: $V = \frac{1}{3}\pi r^2 h$

Substitute $r = \frac{2h}{5}$: $V = \frac{1}{3}\pi \times \left(\frac{2h}{5}\right)^2 \times h$

$$V = \frac{4\pi h^3}{75}$$

Hence: $\frac{dV}{dh} = \frac{4\pi h^2}{25}$

Chain rule: $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

$$\therefore \quad -0.2 = \frac{4\pi h^2}{25} \times \frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{5}{4\pi h^2}$$

When $h = 4$: $\frac{dh}{dt} = -\frac{5}{4\pi \times 16} = -\frac{5}{64\pi}$

$$= -0.0249 \text{ m/min}$$

Thus the water level is decreasing at a rate of 0.0249 m/min.

Alternatively: V and h are both dependent on time, so you can differentiate both sides of the volume equation with respect to time.

$$V = \frac{4\pi h^3}{75}$$

$$\frac{dV}{dt} = \frac{4\pi}{75} \times \frac{d}{dt}\left(h^3\right)$$

Chain rule: $= \frac{4\pi}{75} \times \frac{d}{dh}\left(h^3\right) \times \frac{dh}{dt}$

$$= \frac{4\pi}{75} \times 3h^2 \times \frac{dh}{dt}$$

$\frac{dV}{dt} = -0.2$, so: $-0.2 = \frac{4\pi h^2}{25} \times \frac{dh}{dt}$ as before.

Example 14

If $x = 5t\cos\alpha$ and $y = 5t\sin\alpha - \frac{1}{2}gt^2$, where α and g are constants, find:

(a) the expression for $\frac{dy}{dx}$ as a function of t

(b) the expression for $\frac{dy}{dx}$ when $t = 2$

(c) if $\alpha = \frac{\pi}{4}$ and $g = 9.8$, find the value of $\frac{dy}{dx}$ when $t = 2$.

Solution

(a)

$$\frac{dx}{dt} = 5\cos\alpha$$

$$y = 5t\sin\alpha - \frac{1}{2}gt^2$$

$$\frac{dy}{dt} = 5\sin\alpha - gt$$

$$\frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$$\frac{dy}{dx} = \frac{5\sin\alpha - gt}{5\cos\alpha}$$

(b) $t = 2$

$$\frac{dy}{dx} = \frac{5\sin\alpha - g\times 2}{5\cos\alpha}$$

$$\frac{dy}{dx} = \frac{5\sin\alpha - 2g}{5\cos\alpha}$$

(c) $\alpha = \frac{\pi}{4}, g = 9.8$

$$\frac{dy}{dx} = \frac{5\sin\frac{\pi}{4} - 2\times 9.8}{5\cos\frac{\pi}{4}}$$

$$\frac{dy}{dx} = \frac{\frac{5}{\sqrt{2}} - 19.6}{\frac{5}{\sqrt{2}}}$$

$$\frac{dy}{dx} = -4.544$$

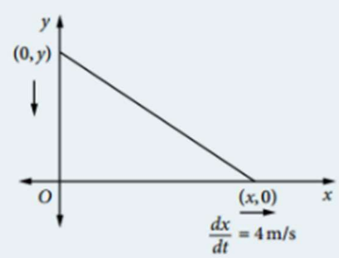
Example 19

A ladder 10 m long has its upper end against a vertical wall and its lower end on a horizontal floor. The lower end is slipping away from the wall at a constant speed of 4 m/s. Find the rate at which the upper end of the ladder is slipping down the wall when the lower end is 6 m from the wall. What is this rate when the upper end is very close to the ground?

Solution

At any time t, the lower end of the ladder is x m from the wall and the upper end is y m above the ground.

Given $\frac{dx}{dt} = 4$, need to find $\frac{dy}{dt}$ when $x = 6$.



By Pythagoras' theorem: $x^2 + y^2 = 100$

$$y = \sqrt{100 - x^2},\ 0 \le x \le 10$$

$$\frac{dy}{dx} = \frac{1}{2}\left(100 - x^2\right)^{-\frac{1}{2}}\times(-2x)$$

$$= \frac{-x}{\sqrt{100 - x^2}},\ 0 \le x < 10$$

For $\frac{dx}{dt} = 4$:

$$\frac{dy}{dt} = \frac{dy}{dx}\times\frac{dx}{dt}$$

$$= \frac{-x}{\sqrt{100 - x^2}}\times 4$$

At $x = 6$:

$$\frac{dy}{dt} = \frac{-4\times 6}{\sqrt{100 - 36}} = -3$$