Exponential functions – the number "*e*"

Exponential functions are of the form $f(x) = a^x$, with a a positive constant.

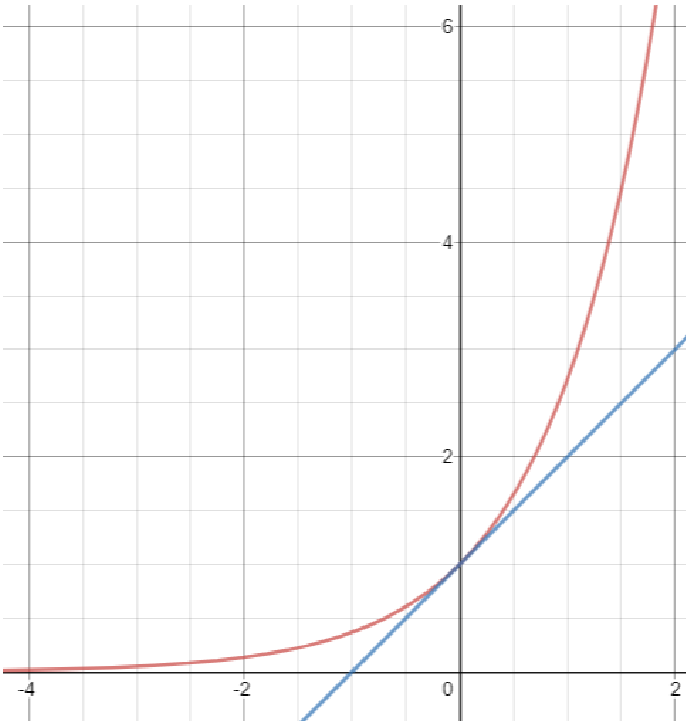
All these functions pass through the point $(0,1)$, as when $x = 0$, $f(0) = a^0 = 1$

The function $f(x) = e^x$ is defined as being the exponential function for which the slope of the tangent at the point $(0,1)$ is 1 i.e.:

$$\lim_{h \to 0} \left(\frac{e^h - e^0}{h} \right) = 1$$

which can also be noted:

$$\lim_{h \to 0} \left(\frac{e^h - 1}{h} \right) = 1$$



The value of the number "*e*" is approximately 2.71828182845…(it never ends, does not repeat, is irrational (i.e. cannot be written as a fraction) and transcendental (i.e. cannot be a solution of a polynomial equation with rational coefficients).

It was named "*e*" after mathematician Leonhard Euler who studied it extensively around beginning of 18th century.

This function $f(x) = e^x$ is called "*the **natural** exponential function*" as of all exponential functions $f(x) = a^x$, with a a positive constant, it is the only one whose gradient at the point $(0,1)$ is 1.

In fact, the number "*e*" was discovered end of 17th century by another mathematician Jacob Bernoulli who was studying compound interest and found that:

$$\lim_{n \to +\infty} \left(1 + \frac{1}{n} \right)^n = e$$

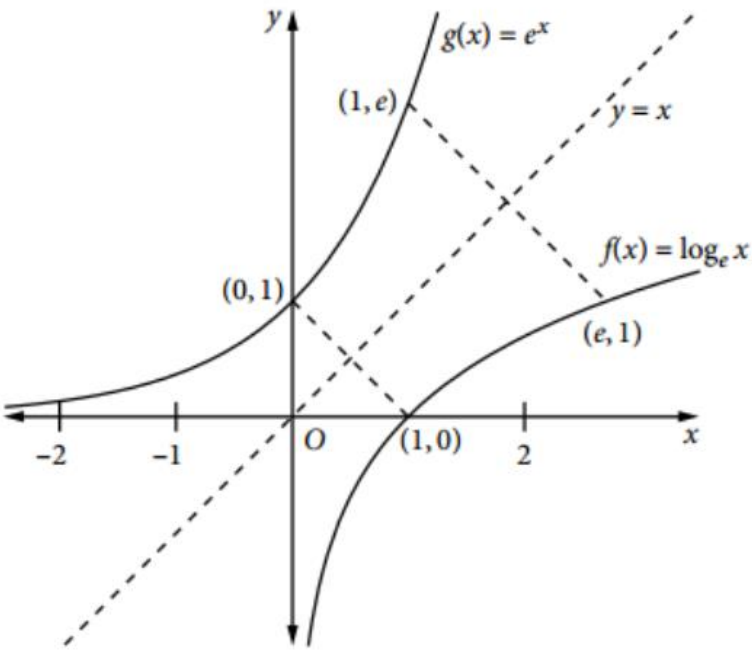
DERIVATIVE OF THE LOGARITHMIC FUNCTION

Summary of previous findings on exponentials and logarithms

Previously we established that:

- $e^0 = 1$ and $\log_e 1 = 0$
- $e^x > 0$ for all x
- the expressions $y = a^x$ and $x = \log_a y$ were equivalent; particularly when $a = e$, we have $x = \log_e y$ which is noted $x = ln\, y$
- $f(x) = a^x$ and $f(x) = \log_a x$ are inverse functions, therefore $a^{\log_a x} = x$ and particularly $e^{ln\, x} = x$
- the domain of $f(x) = \log_a x$ is $x > 0$
- $\log_a xy = \log_a x + \log_a y$ and $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$
- $\log_a x^n = n \log_a x$
- $\log_a x = \frac{\log_b x}{\log_b a}$ change of base rule
- $a^x = e^{x \ln a}$ (that can be proven by taking the log on both sides, which gives $\ln a^x = \ln e^{x \ln a}$ which simplifies as $x \ln a = x \ln a$, which is true therefore the statement $a^x = e^{x \ln a}$ must also be true)

For memory, the diagram below shows the graphs of $f(x) = e^x$ and $f(x) = ln\, x$



Being the inverses of each other, the two graphs are symmetrical with regard to the line $y = x$

DERIVATIVE OF THE LOGARITHMIC FUNCTION

Derivative of $f(x) = e^x$

To find the derivative of $f(x) = e^x$, we go back to the first principle of differentiation:

$$f'(x) = \lim_{h \to 0} \left(\frac{e^{x+h} - e^x}{h} \right)$$

$$f'(x) = \lim_{h \to 0} \left(\frac{e^x e^h - e^x}{h} \right)$$

$$f'(x) = \lim_{h \to 0} \left(\frac{e^x (e^h - 1)}{h} \right)$$

$$f'(x) = \lim_{h \to 0} e^x \left(\frac{e^h - 1}{h} \right)$$

We can replace 1 by e^0 as: $1 = e^0$

$$f'(x) = e^x \times \lim_{h \to 0} \left(\frac{e^h - e^0}{h} \right)$$

But $\left(\frac{e^h - e^0}{h} \right)$ is the slope of the function $f(x) = e^x$ at $x = 0$, which is equal to 1 (by definition of "e")
and therefore: $\lim_{h \to 0} \left(\frac{e^h - e^0}{h} \right) = 1$

Therefore: $f'(x) = e^x \times 1$

$$f'(x) = e^x$$

So the derivative of $f(x) = e^x$ is itself, i.e. $f'(x) = e^x$

<u>This is the only function which is equal to its derivative.</u>

Derivative of $f(x) = \ln x$

$$\frac{d(x)}{dx} = 1$$

But $e^{\ln x} = x$ so:

$$\frac{d(e^{\ln x})}{dx} = 1 \qquad \text{Equation (1)}$$

We know that $\frac{d(e^{f(x)})}{dx} = e^{f(x)} \times \frac{df(x)}{dx}$ (chain rule for differentiation applied to $e^{f(x)}$), so:

$$\frac{d(e^{\ln x})}{dx} = e^{\ln x} \times \frac{d(\ln x)}{dx} = x \times \frac{d(\ln x)}{dx}$$

Therefore Equation (1) becomes:

$$x \times \frac{d(\ln x)}{dx} = 1$$

In conclusion: $$\frac{d(\ln x)}{dx} = \frac{1}{x}$$

DERIVATIVE OF THE LOGARITHMIC FUNCTION

Derivative of $f(x) = e^{ax}$ (where a is a constant)

$$f(x) = e^{ax} = g[h(x)]$$

To calculate this derivative, we use the chain rule as it is a composition of functions.

$$g(X) = e^X \qquad\qquad h(x) = ax$$

$$g'(X) = e^X \qquad\qquad h'(x) = a$$

Therefore, using the chain rule:

$$f'(x) = g'[h(x)] \times h'(x)$$

$$f'(x) = e^{ax} \times a$$

$$f'(x) = a\, e^{ax}$$

Example: if $f(x) = e^{-3x}$ then $f'(x) = -3\, e^{-3x}$

Derivative of $f(x) = ln(ax)$ (where a is a constant)

$$f(x) = ln(ax) = g[h(x)]$$

To calculate this derivative, we use the chain rule as it is a composition of functions.

$$g(X) = ln(X) \qquad\qquad h(x) = ax$$

$$g'(X) = \frac{1}{X} \qquad\qquad h'(x) = a$$

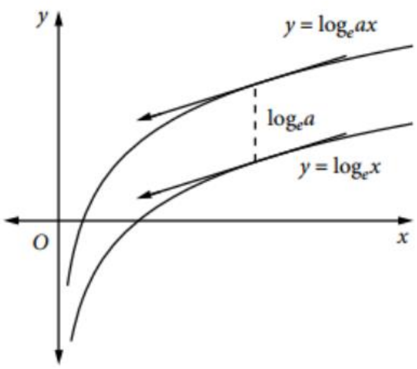
Therefore, using the chain rule:

$$f'(x) = g'[h(x)] \times h'(x)$$

$$f'(x) = \frac{1}{ax} \times a$$

$$f'(x) = \frac{1}{x}$$

The derivatives of $f(x) = ln\,x$ and of $f(x) = ln(ax)$ are both $\frac{1}{x}$, so the only difference between the graphs of the functions is a vertical translation of $\ln a$, as shown on the graph below:



DERIVATIVE OF THE LOGARITHMIC FUNCTION

Derivative of $f(x) = a^x$

As demonstrated before, $a^x = e^{x \ln a}$, therefore $f(x) = e^{x \ln a} = g[h(x)]$

To calculate this derivative, we use the chain rule as it is a composition of functions.

$g(X) = e^X$ $\quad\quad$ $h(x) = x \ln a$

$g'(X) = e^X$ $\quad\quad$ $h'(x) = \ln a$

Therefore, using the chain rule:

$$f'(x) = g'[h(x)] \times h'(x)$$

$$f'(x) = e^{x \ln a} \times \ln a$$

$$f'(x) = a^x \times \ln a$$

Derivative of $f(x) = \log_a x$

The change of base rule states that $\log_a x = \frac{\log_b x}{\log_b a}$; particularly for $b = e$, we obtain:

$$\log_a x = \frac{\ln x}{\ln a}$$

therefore $f(x) = \frac{\ln x}{\ln a} = \frac{1}{\ln a} \times \ln x$

$\frac{1}{\ln a}$ is a constant, therefore: $\quad f'(x) = \frac{1}{\ln a} \times \frac{1}{x} \quad$ or $\quad f'(x) = \frac{1}{x \ln a}$

Example 8

Differentiate:

(a) $\log_e(x^3 + 1)$

(b) $\log_e(x^2 + 2x - 1)$

Solution

(a) Let $y = \log_e(x^3 + 1)$

$= \log_e u$, where $u = x^3 + 1$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{1}{u} \times 3x^2$$

$$= \frac{3x^2}{x^3 + 1}$$

(b) Let $y = \log_e(x^2 + 2x - 1)$

$= \log_e u$, where $u = x^2 + 2x - 1$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{1}{u} \times (2x + 2)$$

$$= \frac{2x + 2}{x^2 + 2x - 1}$$

Example 7

Differentiate with respect to x:

(a) $x^2\log_e(2x)$ **(b)** $\dfrac{\log_e 3x}{e^x}$ **(c)** $f(x)=\dfrac{\log_e x}{\tan x}$

Solution

(a) Let $y=x^2\log_e(2x)=uv$, where $u=x^2$ and $v=\log_e(2x)$

$$\frac{dy}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}$$

$$=2x\log_e 2x+x^2\times\frac{1}{x},\ x>0$$

$$=x(2\log_e 2x+1)$$

(b) Let $y=\dfrac{\log_e 3x}{e^x}=\dfrac{u}{v}$, where $u=\log_e(3x)$ and $v=e^x$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx}=\frac{e^x\times\frac{1}{x}-\log_e 3x\times e^x}{e^{2x}}$$

$$=\frac{e^x(1-x\log_e 3x)}{xe^{2x}}$$

$$=\frac{1-x\log_e 3x}{xe^x}$$

(c) $f(x)=\dfrac{\log_e x}{\tan x}$

$$f'(x)=\frac{\frac{1}{x}\times\tan x-\sec^2 x\log_e x}{\tan^2 x}$$

$$=\frac{\tan x-x\sec^2 x\log_e x}{x\tan^2 x}$$

Example 9

Use the logarithm laws and then find the derivative of each function.

(a) $y=\log_e\left(\dfrac{x^2+1}{x}\right)$ **(b)** $f(x)=\log_e\left(e^x\left(x^2+3\right)\right)$ **(c)** $g(x)=\log_e\left(\dfrac{x^2(e^x-1)}{e^{-x}+1}\right)$

Solution

(a) $y=\log_e\left(\dfrac{x^2+1}{x}\right)=\log_e\left(x^2+1\right)-\log_e x$

$$\frac{dy}{dx}=\frac{2x}{x^2+1}-\frac{1}{x}$$

If a stationary point had to be found then you would write this answer as a single fraction, otherwise leave it. It is a good idea to practice this algebraic simplification before it is needed.

$$\frac{dy}{dx}=\frac{2x^2-(x^2+1)}{x(x^2+1)}=\frac{x^2-1}{x(x^2+1)}$$

(b) $f(x)=\log_e\left(e^x\left(x^2+3\right)\right)$

$$=\log_e e^x+\log_e(x^2+3)$$

$$=x+\log_e(x^2+3)$$

$$f'(x)=1+\frac{2x}{x^2+3}$$

$$=\frac{x^2+3+2x}{x^2+3}$$

(c) $g(x)=\log_e\left(\dfrac{x^2(e^x-1)}{e^{-x}+1}\right)$

$$=\log_e x^2+\log_e(e^x-1)-\log_e(e^{-x}+1)$$

$$=2\log_e x+\log_e(e^x-1)-\log_e(e^{-x}+1)$$

$$g'(x)=\frac{2}{x}+\frac{e^x}{e^x-1}-\frac{e^{-x}}{e^{-x}+1}$$

$$=\frac{2\left(e^x-1\right)\left(e^{-x}+1\right)+xe^x\left(e^{-x}+1\right)-xe^{-x}\left(e^x-1\right)}{x\left(e^x-1\right)\left(e^{-x}+1\right)}$$

$$=\frac{2\left(1+e^x-e^{-x}-1\right)+x+xe^x-x+xe^{-x}}{x\left(e^x-1\right)\left(e^{-x}+1\right)}$$

$$=\frac{2e^x-2e^{-x}+x+xe^x-x+xe^{-x}}{x\left(e^x-1\right)\left(e^{-x}+1\right)}$$