DERIVATIVE OF THE FUNCTION $f(x) = x^n$

If $f(x) = x^n$ then $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$

But: $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + a^2b^{n-3} + ab^{n-2} + b^{n-1})$

Proof:

	a^{n-1}	$a^{n-2}b$	$a^{n-3}b^2$	…	…	a^2b^{n-3}	ab^{n-2}	b^{n-1}
a	a^n	~~$a^{n-1}b$~~	~~$a^{n-2}b^2$~~	…	…	~~a^3b^{n-3}~~	~~a^2b^{n-2}~~	~~ab^{n-1}~~
$-b$	~~$-a^{n-1}b$~~	~~$-a^{n-2}b^2$~~	~~$-a^{n-3}b^3$~~	…	…	~~$-a^2b^{n-2}$~~	~~$-ab^{n-1}$~~	$-b^n$

(the terms of same colour cancel each other)

Therefore:

$$f'(x) = \lim_{h\to 0} \frac{(x+h-x)[(x+h)^{n-1} + (x+h)^{n-2}x + (x+h)^{n-3}x^2 + \ldots + (x+h)^2x^{n-3} + (x+h)x^{n-2} + x^{n-1}]}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{h[(x+h)^{n-1} + (x+h)^{n-2}x + (x+h)^{n-3}x^2 + \ldots + (x+h)^2x^{n-3} + (x+h)x^{n-2} + x^{n-1}]}{h}$$

$$f'(x) = \lim_{h\to 0} [(x+h)^{n-1} + (x+h)^{n-2}x + (x+h)^{n-3}x^2 + \ldots + (x+h)^2x^{n-3} + (x+h)x^{n-2} + x^{n-1}]$$

$$f'(x) = \lim_{h\to 0} \{[x^{n-1} + h\varepsilon_1(x)] + [x^{n-1} + h\varepsilon_2(x)] + [x^{n-1} + h\varepsilon_3(x)] + \ldots + [x^{n-1} + h\varepsilon_{n-3}(x)] + [x^{n-1} + h\varepsilon_{n-2}(x)] + [x^{n-1}]\}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n$ are all functions of x.

$$f'(x) = \lim_{h\to 0} \{n \times x^{n-1} + h \times [\varepsilon_1(x) + \varepsilon_2(x) + \varepsilon_3(x) + \cdots + \varepsilon_n(x)]\}$$

So $f'(x) = nx^{n-1}$ as the term $h \times [\varepsilon_1(x) + \varepsilon_2(x) + \varepsilon_3(x) + \cdots + \varepsilon_n(x)]$ tends towards 0.

Therefore if $f(x) = x^n$ then $f'(x) = nx^{n-1}$

Example: if $f(x) = x^6$ then $f'(x) = 6x^5$

This rule can be extended when n is NOT an integer.

Example: if $f(x) = \sqrt[7]{x} = x^{\frac{1}{7}}$ then $f'(x) = \frac{1}{7}x^{\frac{1}{7}-1} = \frac{1}{7}x^{\frac{-6}{7}} = \frac{1}{7x^{\frac{6}{7}}} = \frac{1}{7\sqrt[7]{x^6}}$

DERIVATIVE OF A FUNCTION MULTIPLIED BY A CONSTANT

DERIVATIVE OF THE SUM OF TWO FUNCTIONS

Derivative of a function multiplied by a constant $\boldsymbol{f(x) = k\, g(x)}$

If $f(x) = k\, g(x)$ where k is a constant

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{k\, g(x+h) - k\, g(x)}{h} = k \times \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = k \times g'(x)$$

Therefore if $f(x) = k \times g(x)$ then $f'(x) = k \times g'(x)$

Example: if $f(x) = 3x$ then $f'(x) = 3$

Derivative of the sum of two functions $\boldsymbol{f(x) = g(x) + h(x)}$

If $f(x) = g(x) + h(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[g(x+h) + h(x+h)] - [g(x) + h(x)]}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x) + h(x+h) - h(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h \to 0} \frac{h(x+h) - h(x)}{h}$$

Therefore if $f(x) = g(x) + h(x)$ then $f'(x) = g'(x) + h'(x)$

Example: if $f(x) = 4x + 2$ then $f'(x) = (4x)' + (2)' = 4 + 0 = 4$