DERIVATIVE OF QUOTIENT OF TWO FUNCTIONS (“quotient rule”)

If $f(x) = \frac{u(x)}{v(x)}$ then $f'(x) = \frac{u'(x)\, v(x) - v'(x)\, u(x)}{[v(x)]^2}$

or noted in a simplified way: $\left(\frac{u}{v}\right)' = \frac{u'\, v - v'\, u}{v^2}$

Proof:

Let $f(x) = \frac{u(x)}{v(x)}$ By definition: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$f'(x) = \lim_{h \to 0} \left\{ \frac{\frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}}{h} \right\}$$

$$f'(x) = \lim_{h \to 0} \frac{1}{h} \left\{ \frac{u(x+h)v(x) - u(x)\ v(x+h)}{v(x+h)v(x)} \right\}$$

We subtract and add the term $u(x)\ v(x)$ to the numerator.

$$f'(x) = \lim_{h \to 0} \frac{1}{h} \left\{ \frac{u(x+h)v(x) - u(x)v(x) + u(x)v(x) - u(x)\ v(x+h)}{v(x+h)v(x)} \right\}$$

$$f'(x) = \lim_{h \to 0} \frac{1}{h} \left\{ v(x) \left[\frac{u(x+h) - u(x)}{v(x+h)v(x)} \right] - u(x) \left[\frac{v(x+h) - v(x)}{v(x+h)v(x)} \right] \right\}$$

$$f'(x) = \lim_{h \to 0} \frac{1}{v(x+h)v(x)} \left\{ v(x) \left[\frac{u(x+h) - u(x)}{h} \right] - u(x) \left[\frac{v(x+h) - v(x)}{h} \right] \right\}$$

$$f'(x) = \lim_{h \to 0} \frac{1}{v(x+h)v(x)} \left\{ v(x) \left[\frac{u(x+h) - u(x)}{h} \right] \right\} - \lim_{h \to 0} \frac{1}{v(x+h)v(x)} \left\{ u(x) \left[\frac{v(x+h) - v(x)}{h} \right] \right\}$$

$f'(x) = \frac{v(x)}{[v(x)]^2} \lim_{h \to 0} \left[\frac{u(x+h) - u(x)}{h} \right] - \frac{u(x)}{[v(x)]^2} \lim_{h \to 0} \left[\frac{v(x+h) - v(x)}{h} \right]$ as $\lim_{h \to 0} v(x+h) = v(x)$

Therefore: $f'(x) = \frac{v(x)}{[v(x)]^2} u'(x) - \frac{u(x)}{[v(x)]^2} v'(x)$

Therefore: $$f'(x) = \frac{u'(x)\, v(x) - v'(x)\, u(x)}{[v(x)]^2}$$

In simplified notation: $\left(\frac{u}{v}\right)' = \frac{u'\, v - v'\, u}{v^2}$ referred to as the “*quotient rule*”

Example: if $f(x) = \frac{3x-7}{x^2+1}$ let: $u(x) = 3x - 7$ and $v(x) = x^2 + 1$

then : $u'(x) = 3$ and $v'(x) = 2x + 1$

and therefore: $f'(x) = \frac{3(x^2+1) - (2x+1)(3x-7)}{(x^2+1)^2}$ which simplifies as: $f'(x) = \frac{-3x^2+11x+}{(x^2+1)^2}$