INDEFINITE INTEGRALS AND SUBSTITUTION

1 Find: **(a)** $\int 2x(x^2-1)^4\,dx$ using the substitution $u = x^2 - 1$

(b) $\int 3x^2(x^3+4)^3\,dx$ using the substitution $u = x^3 + 4$

(c) $\int x^2\sqrt{x^3+1}\,dx$ using the substitution $u = x^3 + 1$.

a) $u = x^2 - 1$, $\therefore \frac{du}{dx} = 2x$ or $du = 2x\,dx$, so:

$$\int (x^2-1)^4 \times 2x\,dx = \int u^4\,du = \frac{u^5}{5} + C = \frac{(x^2-1)^5}{5} + C$$

b) $u = x^3 + 4$, $\therefore \frac{du}{dx} = 3x^2$ or $du = 3x^2\,dx$, so

$$\int (x^3+4)^3 \times 3x^2\,dx = \int u^3 \times du = \frac{u^4}{4} + C = \frac{(x^3+4)^4}{4} + C$$

c) $u = x^3 + 1$, $\therefore \frac{du}{dx} = 3x^2$ or $du = 3x^2 dx$

so $x^2 dx = \frac{1}{3}\,du$

$$\therefore \int x^2\sqrt{x^3+1}\,dx = \int \sqrt{x^3+1} \times x^2 dx$$

$$= \int \sqrt{u} \times \frac{1}{3}\,du$$

$$= \frac{1}{3}\int u^{1/2}\,du$$

$$= \frac{1}{3}\,\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$= \frac{1}{3} \times \frac{u^{3/2}}{\frac{3}{2}} + C = \frac{2}{9}u^{3/2} + C = \frac{2}{9}(x^3+1)^{3/2} + C$$

INDEFINITE INTEGRALS AND SUBSTITUTION

2 Find: **(a)** $\int (2t+1)^3\,dt$ using the substitution $u = 2t + 1$

(b) $\int \frac{2x}{\sqrt{x^2-4}}\,dx$ using the substitution $u = x^2 - 4$

(c) $\int (2x+1)\left(x^2+x+2\right)^5 dx$ using the substitution $u = x^2 + x + 2$.

a) $u = 2t+1$ so $\frac{du}{dt} = 2$ $du = 2dt$ or $dt = \frac{1}{2}du$

$\int (2t+1)^3 dt = \int u^3 \times \frac{1}{2}du = \frac{1}{2}\int u^3 du = \frac{1}{2} \times \frac{u^4}{4} + C = \frac{(2t+1)^4}{8} + C$

b) $u = x^2 - 4$ so $\frac{du}{dx} = 2x$ or $du = 2x\,dx$

$\int \frac{2x}{\sqrt{x^2-4}}dx = \int \frac{du}{\sqrt{u}} = \int u^{-1/2}du = \frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = \frac{u^{1/2}}{1/2} + C = 2\sqrt{u} + C$

$= 2\sqrt{x^2-4} + C$

c) $u = x^2 + x + 2$ so $\frac{du}{dx} = 2x+1$ or $du = (2x+1)dx$

$\int (x^2+x+2)^5 (2x+1)\,dx = \int u^5 \times du = \frac{u^6}{6} + C$

$= \frac{(x^2+x+2)^6}{6} + C$

INDEFINITE INTEGRALS AND SUBSTITUTION

4 Find: **(a)** $\int (3-2x)^6\,dx$ using the substitution $u = 3-2x$

(b) $\int \frac{3x+1}{(3x^2+2x+5)^2}\,dx$ using the substitution $u = 3x^2+2x+5$

(c) $\int (x^2-2x)(x^3-3x^2+1)^4\,dx$ using the substitution $u = x^3-3x^2+1$.

a) $u = 3-2x$ so $\frac{du}{dx} = -2$ or $du = -2\,dx$ or $dx = -\frac{1}{2}\,du$

$$\int (3-2x)^6\,dx = \int u^6 \times \left(-\frac{1}{2}\,du\right) = -\frac{1}{2}\int u^6\,du = -\frac{1}{2}\times\frac{u^7}{7} + C = \frac{-(3-2x)^7}{14} + C$$

b) $u = 3x^2+2x+5$ so $\frac{du}{dx} = 6x+2 = 2(3x+1)\,dx$

so $(3x+1)\,dx = \frac{1}{2}\,du$

$$\int \frac{3x+1}{(3x^2+2x+5)^2}\,dx = \int \frac{1}{u^2}\times\frac{1}{2}\,du = \frac{1}{2}\int\frac{du}{u^2} = \frac{1}{2}\int u^{-2}\,du$$

$$\text{——} = \frac{1}{2}\,\frac{u^{-2+1}}{(-2+1)} + C = \frac{1}{2}\,\frac{u^{-1}}{(-1)} + C = -\frac{u^{-1}}{2} + C = -\frac{1}{2u} + C$$

$$\text{——} = -\frac{1}{2(3x^2+2x+5)} + C$$

c) $u = x^3-3x^2+1$ so $\frac{du}{dx} = 3x^2-6x = 3(x^2-2x)$

$du = 3(x^2-2x)\,dx$

or $(x^2-2x)\,dx = \frac{1}{3}\,du$

$$\int (x^2-2x)(x^3-3x^2+1)^4\,dx = \int \frac{1}{3}\,du \times u^4 = \frac{1}{3}\int u^4\,du = \frac{1}{3}\times\frac{u^5}{5} + C$$

$$\text{——} = \frac{1}{15}u^5 + C = \frac{1}{15}(x^3-3x^2+1)^5 + C$$

INDEFINITE INTEGRALS AND SUBSTITUTION

6 Find: **(a)** $\int 2t\sqrt{1-t^2}\,dt$ using the substitution $u = 1 - t^2$

(b) $\int x\sqrt{a^2 - x^2}\,dx$ using the substitution $u = a^2 - x^2$

(c) $\int z\sqrt[3]{z^2+1}\,dz$ using the substitution $u = z^2 + 1$.

a) $u = 1 - t^2 \qquad \frac{du}{dt} = -2t \qquad$ so $2t\,dt = -du$

$$\int \sqrt{1-t^2} \times 2t\,dt = \int \sqrt{u} \times (-du) = -\int u^{1/2}\,du = -\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$= -\frac{u^{3/2}}{\frac{3}{2}} + C = -\frac{2}{3}u^{3/2} + C = -\frac{2}{3}(1-t^2)^{3/2} + C$$

b) $u = a^2 - x^2, \quad \frac{du}{dx} = -2x \qquad$ so $x\,dx = -\frac{1}{2}du$

$$\int \sqrt{a^2-x^2} \times x\,dx = \int \sqrt{u} \times \left(-\frac{1}{2}du\right) = -\frac{1}{2}\int u^{1/2}\,du = -\frac{1}{2}\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$= -\frac{1}{2}\frac{u^{3/2}}{\frac{3}{2}} + C = -\frac{u^{3/2}}{3} + C = -\frac{(a^2-x^2)^{3/2}}{3} + C$$

c) $u = z^2 + 1 \qquad$ so $\frac{du}{dz} = 2z \qquad$ or $z\,dz = \frac{1}{2}du$

$$\int \sqrt[3]{z^2+1} \times z\,dz = \int \sqrt[3]{u} \times \frac{1}{2}du = \frac{1}{2}\int u^{1/3}\,du = \frac{1}{2}\frac{u^{\frac{1}{3}+1}}{\frac{1}{3}+1} + C$$

$$= \frac{1}{2} \times \frac{u^{4/3}}{\frac{4}{3}} + C = \frac{3}{8}u^{4/3} + C$$

$$\therefore \int z\sqrt[3]{z^2+1}\,dz = \frac{3}{8}(z^2+1)^{4/3} + C$$

INDEFINITE INTEGRALS AND SUBSTITUTION

7 Find: **(a)** $\int y\sqrt{y+1}\,dy$ using the substitution $u = y + 1$

(b) $\int \frac{x}{(x-1)^3}\,dx$ using the substitution $u = x - 1$

(c) $\int \frac{x}{\sqrt{2x-1}}\,dx$ using the substitution $u = 2x - 1$.

a) $u = y+1$ so $\frac{du}{dy} = 1$ or $dy = du$

$$\int y\sqrt{y+1}\,dy = \int (u-1)\sqrt{u}\,du = \int \left(u\sqrt{u} - \sqrt{u}\right) du = \int \left(u^{3/2} - u^{1/2}\right) du$$

$$= \frac{u^{\frac{3}{2}+1}}{\frac{3}{2}+1} - \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{5}(y+1)^{5/2} - \frac{2}{3}(y+1)^{3/2} + C$$

b) $u = x-1$ so $\frac{du}{dx} = 1$ or $du = dx$

$$\int \frac{x}{(x-1)^3}\,dx = \int \frac{u+1}{u^3}\,du = \int \left(\frac{1}{u^2} + \frac{1}{u^3}\right) du = \int \left(u^{-2} + u^{-3}\right) du$$

$$= \frac{u^{-2+1}}{-2+1} + \frac{u^{-3+1}}{-3+1} + C = -\frac{1}{u} - \frac{1}{2u^2} + C$$

$$= -\frac{1}{(x-1)} - \frac{1}{2(x-1)^2} + C$$

c) $u = 2x-1$ so $\frac{du}{dx} = 2$ or $dx = \frac{1}{2}du$

$x = \frac{u+1}{2}$

$$\int \frac{x\,dx}{\sqrt{2x-1}} = \int \frac{\frac{u+1}{2}}{\sqrt{u}} \times \frac{1}{2}\,du = \int \frac{u+1}{4\sqrt{u}}\,du = \frac{1}{4}\int \left(u^{1/2} + u^{-1/2}\right) du$$

$$= \frac{1}{4}\left[\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right] + C = \frac{1}{4}\left(\frac{u^{3/2}}{\frac{3}{2}} + \frac{u^{1/2}}{\frac{1}{2}}\right) + C$$

$$= \frac{1}{6}u^{3/2} + \frac{1}{2}u^{1/2} + C = \frac{1}{6}(2x-1)^{3/2} + \frac{1}{2}(2x-1)^{1/2} + C$$

11 $\frac{dx}{dt} = \frac{t-1}{\sqrt{t^2-2t+4}}$ and $x = 10$ when $t = 0$. Use the substitution $u = t^2 - 2t + 4$ to find x in terms of t.

$$x(t) = \int \frac{t-1}{\sqrt{t^2-2t+4}}\, dt$$

$u = t^2 - 2t + 4$ so $\frac{du}{dt} = 2t - 2 = 2(t-1)$

$du = 2(t-1)\,dt$ or $(t-1)\,dt = \frac{1}{2}\,du$

$$x(t) = \int \frac{\frac{1}{2}\,du}{\sqrt{u}} = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\,\frac{u^{-\frac{1}{2}+1}}{\left(-\frac{1}{2}+1\right)} + C$$

$$x(t) = \frac{1}{2}\cdot\frac{u^{1/2}}{\frac{1}{2}} + C = \frac{u^{1/2}}{1} + C = \sqrt{u} + C$$

$$x(t) = \sqrt{t^2-2t+4} + C$$

when $t = 0$, $x(0) = \sqrt{0^2 - 2\times 0 + 4} + C = 2 + C$

But $x(0) = 10$ so $2 + C = 10$, so $C = 8$

$$x(t) = \sqrt{t^2-2t+4} + 8$$