RANDOM VARIABLES - CHAPTER REVIEW

1 For the uniform continuous variable with probability density function $f(x) = \begin{cases} k, & 0 \le x \le 10 \\ 0, & \text{otherwise} \end{cases}$, find the following values.

(a) k **(b)** $P(X \le 3)$ **(c)** $P(X \le 5)$ **(d)** $P(2 \le X \le 9)$

a) $\int_0^{10} k\,dx = 1$ so $k\int_0^{10} dx = 1 \Leftrightarrow k[x]_0^{10} = 1$ so $10k = 1$

$\therefore k = 1/10$

b) $P(X \leqslant 3) = \int_0^3 \frac{1}{10}\,dx = 3/10$

c) $P(X \leqslant 5) = \int_0^5 \frac{1}{10}\,dx = 5/10 = 1/2$

d) $P(2 \leqslant X \leqslant 9) = \int_2^9 \frac{1}{10}\,dx = \frac{(9-2)}{10} = 7/10$

2 Does the hybrid function $f(x) = \begin{cases} x^2 + 4x - \frac{10}{3}, & 0 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$ represent a probability density function?

To be a pdf, f must 1) be always positive
2) $\int_0^3 f(x)\,dx = 1$

let's check 1): $\Delta = 16 - 4 \times \left(-\frac{10}{3}\right) \times 1 = \frac{88}{3}$ so 2 roots.

$x_1 = \frac{-4 + \sqrt{88/3}}{2} \sim +0.71$

$x_2 = \frac{-4 - \sqrt{8/3}}{2} \sim -2.81$

So $f(x) < 0$ for part of the domain
$\therefore f$ is not a probability density function

~~Now we check 2) $\int_0^3 \left(x^2 + 4x - \frac{10}{3}\right) dx = \left[\frac{x^3}{3} + \frac{4x^2}{2} - \frac{10x}{3}\right]_0^3$~~

~~$\int_0^3 \left(x^2 + 4x - \frac{10}{3}\right) dx = \frac{3^3}{3} + 2 \times 3^2 - \frac{10 \times 3}{3} = 9 + 18 - 10 = 17$~~

~~So $f(x)$ is not a probability density function.~~

3 For the continuous variable with probability density function $f(x) = \begin{cases} \frac{x^2}{12}, & 0 \le x \le \sqrt[3]{36} \\ 0, & \text{otherwise} \end{cases}$, find the exact value of the median.

The median m is the value such that $\int_0^m \frac{x^2}{12}\,dx = \frac{1}{2}$

$$\int_0^m \frac{x^2}{12}\,dx = \frac{1}{12}\left[\frac{x^3}{3}\right]_0^m = \frac{1}{12} \times \frac{m^3}{3} = \frac{m^3}{36}$$

So $\frac{m^3}{36} = \frac{1}{2}$ so $m^3 = \frac{36}{2} = 18$

$$m = \sqrt[3]{18} = \sqrt[3]{2 \times 3^2}$$

4 Honey is packed in jars with a labelled mass of 500 g. The actual amount X g of honey in the jar is such that $X \sim N(500, 25)$.

(a) Find the probability that a jar chosen at random contains less than 495 g.

(b) Jars containing less than 490 g cannot be sold. In a batch of 1000 how many jars would you expect to be rejected?

a) $X \sim N(500, 25)$ so $\sigma^2 = 25$, $\sigma = 5$

So $495 = \mu - \sigma$ $\quad P(X < 495) = \frac{1 - 0.68}{2} = 16\%$

b) $490 = \mu - 2\sigma$ $\quad P(X < 490) = \frac{1 - 0.95}{2} = 0.025$ so 2.5%

$\therefore$ in a batch of 1,000 jars, 25 would be rejected.

5 If $X \sim N(12, 16)$, find the exact z values corresponding to the following x values.

(a) $x = 8$ **(b)** $x = 20$

$\sigma^2 = 16$ so $\sigma = 4$

a) $z = \frac{x-\mu}{\sigma} = \frac{8-12}{4} = -1$

b) $z = \frac{x-\mu}{\sigma} = \frac{20-12}{4} = \frac{8}{4} = 2$

6 If $X \sim N(28, 16)$, find the exact z values corresponding to the following x values.

(a) $x = 24$ **(b)** $x = 40$

$\sigma^2 = 16$, so $\sigma = 4$

a) $z = \frac{x-\mu}{\sigma} = \frac{24-28}{4} = \frac{-4}{4} = -1$

b) $z = \frac{x-\mu}{\sigma} = \frac{40-28}{4} = \frac{12}{4} = 3$

7 If $f(x) = \begin{cases} \frac{2x}{15+k}, & 0 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$ defines a probability density function, then the value of k is:

A $\frac{2}{15}$ **B** -6 **C** $-\frac{2}{15}$ **D** $\frac{1}{5}$

$\int_0^3 f(x)\,dx = 1$, so $\int_0^3 \frac{2x}{15+k}\,dx = \frac{2}{15+k}\int_0^3 x\,dx$

$\int_0^3 \frac{2x\,dx}{15+k} = \frac{2}{15+k}\left[\frac{x^2}{2}\right]_0^3 = \frac{3^2}{15+k} = \frac{9}{15+k}$ which must be equal to 1

so $\frac{9}{15+k} = 1$ so $15+k = 9 \iff k = 9-15 = -6$

Response $\boxed{B}$

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8 $X \sim N(10, 9)$. You would expect about 95% of observations to be in the range:

A −8 to 28 **B** 0 to 28 **C** 10 to 19 **D** 4 to 16

95% of values are in the range $[\mu - 2\sigma, \mu + 2\sigma]$

Here $\sigma^2 = 9$ so $\sigma = 3$

$\therefore$ 95% of values would be in the range $[10 - 2 \times 3, 10 + 2 \times 3]$

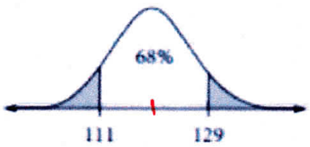
or $[4, 16]$ Response $\boxed{D}$

9 Consider the following graph.

The graph is best described by:

A $N(111, 129)$ **B** $N(120, 81)$

C $N(120, 3)$ **D** $N(120, 9)$



$\mu = \frac{129 + 111}{2} = 120$

Also $\sigma = 129 - 120 = 120 - 111 = 9$ so $\sigma^2 = 81$

$N(120, 81)$ Response $\boxed{B}$

10 The graph shows two normal distributions drawn on the same set of axes. Graph 1 has a marker on it.

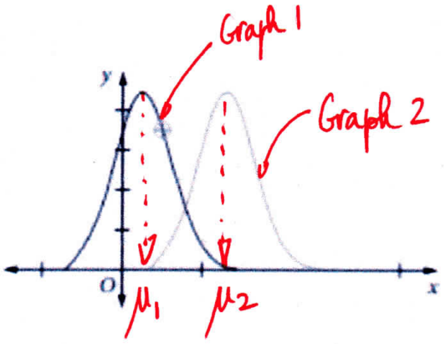
Compared to Graph 1, Graph 2 has:

A the same mean but a larger variation

B the same mean but a smaller variation

$\boxed{\textbf{C}}$ the same variation but a larger mean

D the same variation but a smaller mean



Response $\boxed{C}$

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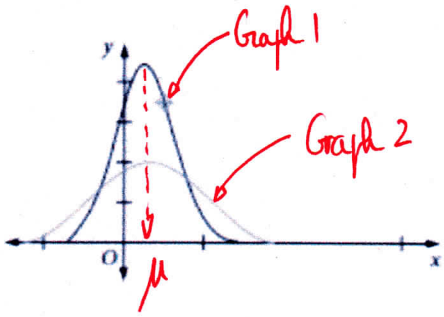


14 For their Mathematics assignment, Xandra and Zack have decided to record the times they spend completing the daily homework. Xandra's times are represented by the continuous random variable, X, in hours, with the probability density function:

$$f(x)=\begin{cases}-\frac{1}{9}\left(x^2-4x\right), & 0\le x\le 3\\ 0, & \text{otherwise}\end{cases}$$

Zack's times are represented by the continuous random variable, Z, in hours, with the probability density function:

$$f(z)=\begin{cases}\frac{2}{57}\left(z^2-9z+20\right), & 0\le z\le 3\\ 0, & \text{otherwise}\end{cases}$$

(a) Calculate the average time each student spends completing homework each day and compare the results.

Xandra has a school concert to attend and has to complete her set homework within 1.5 hours.

(b) Calculate the probability that Xandra will complete her homework in the required time.

(c) Calculate the variance for the two variables.

a) Xandra: $E_{Xandra}(X)=\int_0^3 x\times\left[-\frac{1}{9}\left(x^2-4x\right)\right]dx=-\frac{1}{9}\left[\frac{x^4}{4}-\frac{4x^3}{3}\right]_0^3=1.75$ hours

Zack: $E_{Zack}(X)=\int_0^3 z\times\left[\frac{2}{57}\left(z^2-9z+20\right)\right]dz$

$E_{Zack}(X)=\frac{2}{57}\left[\frac{z^4}{4}-9\frac{z^3}{3}+20\frac{z^2}{2}\right]_0^3=\frac{2}{57}\left[\frac{3^4}{4}-3\times3^3+10\times9\right]=1.03$ hours $=39/38$

So Xandra complete her homework on average in 1.75 hours, compared to Zack in 1.03 hours.

b) $P(X<1.5)=\int_0^{1.5}-\frac{1}{9}\left(x^2-4x\right)dx=-\frac{1}{9}\left[\frac{x^3}{3}-4\frac{x^2}{2}\right]_0^{1.5}$

$P(X<1.5)=-\frac{1}{9}\left(\frac{1.5^3}{3}-2\times1.5^2\right)=0.375$ so 37.5%

c) Xandra: $E(X^2)=\int_0^3 x^2\left[-\frac{1}{9}\left(x^2-4x\right)\right]dx=-\frac{1}{9}\left[\frac{x^5}{5}-\frac{4x^4}{4}\right]_0^3$

$E(X^2)=\frac{1}{9}\left[x^4-\frac{x^5}{5}\right]_0^3=\frac{1}{9}\left(3^4-\frac{3^5}{5}\right)=3.6$

So $V(X)=E(X^2)-\left[E(X)\right]^2=3.6-1.75^2=0.5375$

Zack: $E(X^2)=\int_0^3\frac{2}{57}z^2\left(z^2-9z+20\right)dz=\frac{2}{57}\left[\frac{z^5}{5}-\frac{9z^4}{4}+20\frac{z^3}{3}\right]_0^3$

$E(X^2)=\frac{2}{57}\left[\frac{3^5}{5}-\frac{9\times3^4}{4}+20\times\frac{3^3}{3}\right]=\frac{309}{190}$

so $V(X)=\frac{309}{190}-\left(\frac{39}{38}\right)^2=\frac{4137}{7220}\simeq0.5730$

15 When fishing, fish shorter than a given length have to be thrown back into the water. Red snapper have a minimum allowed length of 30 cm and barramundi have a minimum allowed length of 55 cm. Let the variable *R* represent the lengths of red snapper caught, in cm, and the variable *B* represent the lengths of barramundi caught, in cm.

Observations of caught red snapper show that *R* is normally distributed with mean 36 and a variance of 9. Observations of caught barramundi show that *B* is normally distributed with a variance of 16.

(a) Calculate the mean length of barramundi caught, correct to the nearest whole number, if 2.5% of the barramundi caught are less than 54 cm.

(b) Calculate the *z*-scores for the minimum allowed lengths for both variables, *R* and *B*, and interpret what this means in terms of which of the two species are more likely to be too small to keep.

$R \sim N(36, 9)$ so $\sigma_R^2 = 9$ $\quad \sigma_R = 3$

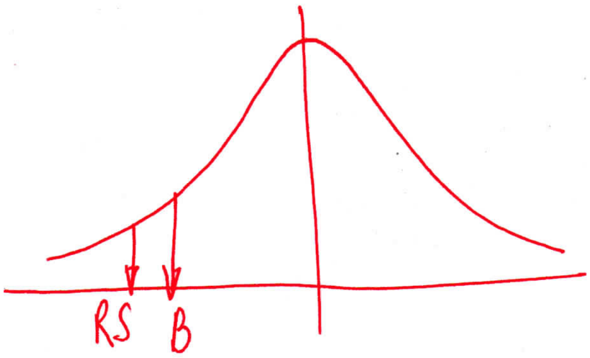
$B \sim N(\mu_B, 16)$ so $\sigma_B^2 = 16$ $\quad \sigma_B = 4$

a) $P(B < \mu - 2\sigma) = 2.5\%$

so $\mu - 2\sigma = 54$ $\quad$ so $\mu = 54 + 2 \times 4 = 62$ cm

b) For Red Snapper $z = \dfrac{30-36}{3} = -2$

For Barramundi $z = \dfrac{55-62}{4} = -1.75$



The proportion of Red snapper that must be thrown back is smaller than the proportion of Barramundi thrown back.