DIFFERENTIATION OF INVERSE TRIGONOMETRIC FUNCTIONS

2 Differentiate the following.

(a) $y = \tan^{-1} 5x$ (b) $y = 3\tan^{-1}(1-x)$ (c) $y = \tan^{-1} x^2$ (d) $y = (\tan^{-1} x)^2$

a) $\frac{d}{dx}\left[\tan^{-1}(f(x))\right] = \frac{f'(x)}{1+[f(x)]^2}$ (taken from formula sheet HSC)

So $\frac{dy}{dx} = \frac{d}{dx}\left[\tan^{-1}(5x)\right] = \frac{5}{1+(5x)^2} = \frac{5}{1+25x^2}$

b) $\frac{d}{dx}\left[3\tan^{-1}(1-x)\right] = 3\frac{d}{dx}\left[\tan^{-1}(1-x)\right] = 3 \times \frac{(-1)}{1+(1-x)^2}$

$\quad = \frac{-3}{x^2-2x+2}$

c) $\frac{d}{dx}\left[\tan^{-1}x^2\right] = \frac{2x}{1+(x^2)^2} = \frac{2x}{1+x^4}$

d) $\frac{d}{dx}\left[(\tan^{-1}x)^2\right] = 2\tan^{-1}x \times \frac{d}{dx}(\tan^{-1}x)$ (chain rule)

$\quad = 2\tan^{-1}x \times \frac{1}{1+x^2}$

$\quad = \frac{2\tan^{-1}x}{1+x^2}$

DIFFERENTIATION OF INVERSE TRIGONOMETRIC FUNCTIONS

2 Differentiate the following.

(i) $y = \sin^{-1}\left(\frac{x}{4}\right)$ (j) $y = 2\cos^{-1}\left(\frac{3x}{2}\right)$ (k) $y = \log_e(\sin^{-1}x)$ (l) $y = \log_e(\cos^{-1}2x)^2$

i) $\frac{d}{dx}\left[\sin^{-1}(f(x))\right] = \frac{f'(x)}{\sqrt{1-[f(x)]^2}}$ (from HSC formula sheet)

$$\frac{d}{dx}\left[\sin^{-1}\left(\frac{x}{4}\right)\right] = \frac{1/4}{\sqrt{1-\left(\frac{x}{4}\right)^2}} = \frac{1}{4\sqrt{1-\left(\frac{x}{4}\right)^2}} = \frac{1}{\sqrt{16-x^2}}$$

j) $\frac{d}{dx}\left[\cos^{-1}(f(x))\right] = -\frac{f'(x)}{\sqrt{1-[f(x)]^2}}$ (from HSC formula sheet)

$$\frac{d}{dx}\left[2\cos^{-1}\left(\frac{3x}{2}\right)\right] = 2\frac{d}{dx}\left[\cos^{-1}\left(\frac{3x}{2}\right)\right] = 2\times\left(-\frac{3/2}{\sqrt{1-\left(\frac{3x}{2}\right)^2}}\right)$$

$$\_\_\_ = -\frac{3}{\frac{1}{2}\sqrt{4-9x^2}} = -\frac{6}{\sqrt{4-9x^2}}$$

k) $\frac{d}{dx}\left[\ln(\sin^{-1}x)\right] = \frac{1}{\sin^{-1}x}\times\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sin^{-1}x}\times\frac{1}{\sqrt{1-x^2}}$

So $\frac{d}{dx}\left[\ln(\sin^{-1}x)\right] = \frac{1}{\sin^{-1}x\times\sqrt{1-x^2}}$

l) $\frac{d}{dx}\left[\ln(\cos^{-1}2x)^2\right] = \frac{d}{dx}\left[2\ln(\cos^{-1}2x)\right] = 2\frac{d}{dx}\left[\ln(\cos^{-1}2x)\right]$

$$\_\_\_ = 2\times\frac{1}{\cos^{-1}2x}\times\left(\frac{-2}{\sqrt{1-(2x)^2}}\right)$$

$$\_\_\_ = \frac{-4}{\cos^{-1}2x\sqrt{1-4x^2}}$$

DIFFERENTIATION OF INVERSE TRIGONOMETRIC FUNCTIONS

2 Differentiate the following.

(q) $y=\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ **(r)** $y=\cos^{-1}x+\cos^{-1}(-x)$ **(s)** $y=\tan x\tan^{-1}x$ **(t)** $y=\tan^{-1}\left(\sqrt{x^2-1}\right)$

q) $\dfrac{d}{dx}\left[\dfrac{1-x^2}{1+x^2}\right]=\dfrac{-2x(1+x^2)-2x(1-x^2)}{(1+x^2)^2}=\dfrac{-4x}{(1+x^2)^2}$, so:

$$\frac{d}{dx}\left[\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]=\frac{\frac{4x}{(1+x^2)^2}}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}}=\frac{4x}{(1+x^2)^2\sqrt{1-\frac{(1-x^2)^2}{(1+x^2)^2}}}$$

$$\text{———}=\frac{4x}{(1+x^2)\sqrt{(1+x^2)^2-(1-x^2)^2}}=\frac{4x}{(1+x^2)\sqrt{4x^2}}$$

$$\text{———}=\frac{2}{1+x^2}$$

r) $\dfrac{d}{dx}\left[\cos^{-1}x+\cos^{-1}(-x)\right]=\dfrac{d}{dx}\left[\cos^{-1}x\right]+\dfrac{d}{dx}\left[\cos^{-1}(-x)\right]=\dfrac{-1}{\sqrt{1-x^2}}-\dfrac{(-1)}{\sqrt{1-(-x)^2}}=0$

s) $\dfrac{d}{dx}\left[\tan x\times\tan^{-1}x\right]=\sec^2x\times\tan^{-1}x+\tan x\times\dfrac{1}{1+x^2}$

t) $\dfrac{d}{dx}\left[\tan^{-1}\left(\sqrt{x^2-1}\right)\right]=\dfrac{\frac{x}{\sqrt{x^2-1}}}{1+(x^2-1)}$

as $\dfrac{d}{dx}\left[\sqrt{x^2-1}\right]=\dfrac{d}{dx}\left[(x^2-1)^{1/2}\right]=\dfrac{1}{2}(x^2-1)^{1/2-1}\times 2x=\dfrac{x}{\sqrt{x^2-1}}$

$$\therefore\ \frac{d}{dx}\left[\tan^{-1}\left(\sqrt{x^2-1}\right)\right]=\frac{\frac{x}{\sqrt{x^2-1}}}{x^2}=\frac{\frac{1}{\sqrt{x^2-1}}}{x}$$

$$\text{———}=\frac{1}{x\sqrt{x^2-1}}$$

DIFFERENTIATION OF INVERSE TRIGONOMETRIC FUNCTIONS

5 If $y = \cos^{-1} x + \cos^{-1}(-x)$, find $\frac{dy}{dx}$ and show that $y = \pi$ for all x in the domain.

as per Question 2, $\frac{d}{dx}\left[\cos^{-1}x + \cos^{-1}(-x)\right] = 0$

$\therefore$ The function $f(x) = \cos^{-1}x + \cos^{-1}(-x)$ is a constant.

We take a value of x at random, e.g. $x = \frac{\sqrt{2}}{2}$

$f\left(\frac{\pi}{4}\right) = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) + \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4} + \frac{3\pi}{4} = \pi$

$\therefore$ $f(x) = \pi$ on all its domain, which is $[-1, 1]$

6 If $y = \sin^{-1} x + \sin^{-1}(-x)$, find $\frac{dy}{dx}$ and show that $y = 0$ for all x in the domain.

$$\frac{d}{dx}\left[\sin^{-1}x + \sin^{-1}(-x)\right] = \frac{d}{dx}\left[\sin^{-1}x\right] + \frac{d}{dx}\left[\sin^{-1}(-x)\right]$$

$$= \frac{1}{\sqrt{1-x^2}} + \frac{(-1)}{\sqrt{1-(-x)^2}} = 0$$

$\therefore$ $\sin^{-1}x + \sin^{-1}(-x) =$ Constant on all its domain which is $[-1, 1]$

For example, for $x = 0$

$$\sin^{-1}x + \sin^{-1}(-x) = \sin^{-1}0 + \sin^{-1}0 = 0 + 0 = 0$$

$\therefore$ Constant $= 0$ and $\sin^{-1}x + \sin^{-1}(-x) = 0$

DIFFERENTIATION OF INVERSE TRIGONOMETRIC FUNCTIONS

7 **(a)** Differentiate $x\tan^{-1}x$. **(b)** Hence find $\int \tan^{-1}x\,dx$.

(c) Use the substitution $u = \log_e x$ to evaluate $\int_1^e \frac{\tan^{-1}(\log_e x)}{x}dx$.

a) Product rule

$$\frac{d}{dx}\left[x\tan^{-1}x\right] = \tan^{-1}x + x \times \frac{d}{dx}\left(\tan^{-1}x\right) = \tan^{-1}x + x \times \frac{1}{1+x^2}$$

$$= \tan^{-1}x + \frac{x}{1+x^2}$$

b) $\therefore$ from a) $\tan^{-1}x = \frac{d}{dx}\left[x\tan^{-1}x\right] - \frac{x}{1+x^2}$

$$\int \tan^{-1}x\,dx = \int\left[\frac{d}{dx}(x\tan^{-1}x) - \frac{x}{1+x^2}\right]dx$$

$$= \int \frac{d}{dx}(x\tan^{-1}x) - \int\frac{x}{1+x^2}dx$$

$$= x\tan^{-1}x + C - \int\frac{x}{1+x^2}dx$$

Let $u = 1+x^2$ so $\frac{du}{dx} = 2x$ or $x\,dx = \frac{1}{2}du$

$$= x\tan^{-1}x + C - \frac{1}{2}\int\frac{du}{u} = x\tan^{-1}x + C - \frac{1}{2}\left[\ln|u|\right]$$

$$= x\tan^{-1}x + C - \frac{1}{2}\ln(1+x^2)$$

c) $u = \ln x$ $\therefore \frac{du}{dx} = \frac{1}{x}$ so $du = \frac{dx}{x}$

$$\int_1^e \frac{\tan^{-1}(\ln x)}{x}dx = \int_0^1 \tan^{-1}(u)\,du = \left[u\tan^{-1}u - \frac{1}{2}\ln(1+u^2)\right]_0^1$$

$$= \left(\tan^{-1}1 - \frac{1}{2}\ln 2\right) - (0-0) = \tan^{-1}1 - \ln\sqrt{2}$$

$$= \frac{\pi}{4} - \ln\sqrt{2}$$

DIFFERENTIATION OF INVERSE TRIGONOMETRIC FUNCTIONS

8 **(a)** State the domain of $f(x) = \tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)$.

(b) Find $f'(x)$. **(c)** Find $f(1)$ and $f(-1)$. **(d)** Sketch the graph of $y = f(x)$.

a) The domain of $y = \tan^{-1}x$ is $(-\infty, +\infty)$

For $\tan^{-1}\left(\frac{1}{x}\right)$, x must be different of 0.

$\therefore$ the domain of $f(x)$ is $\mathbb{R} - \{0\}$.

b) $$\frac{d}{dx}\left[\tan^{-1}x + \tan^{-1}\left(\frac{1}{x}\right)\right] = \frac{d}{dx}\left[\tan^{-1}x\right] + \frac{d}{dx}\left[\tan^{-1}\left(\frac{1}{x}\right)\right]$$

$$= \frac{1}{1+x^2} + \frac{\left(-1/x^2\right)}{1 + \left(1/x\right)^2} = \frac{1}{1+x^2} - \frac{1}{x^2+1} = 0$$

As $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}\left(x^{-1}\right) = (-1)\, x^{-1-1} = -1/x^2$

Therefore $f(x) = \tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right)$ is a constant as its derivative is 0, on its domain of definition.

c) $f(1) = \tan^{-1}(1) + \tan^{-1}\left(\frac{1}{1}\right) = 2\tan^{-1}(1) = 2 \times \frac{\pi}{4} = \frac{\pi}{2}$

$f(-1) = \tan^{-1}(-1) + \tan^{-1}\left(\frac{1}{(-1)}\right) = 2\tan^{-1}(-1) = 2 \times \left(\frac{-\pi}{4}\right) = -\frac{\pi}{2}$

d)