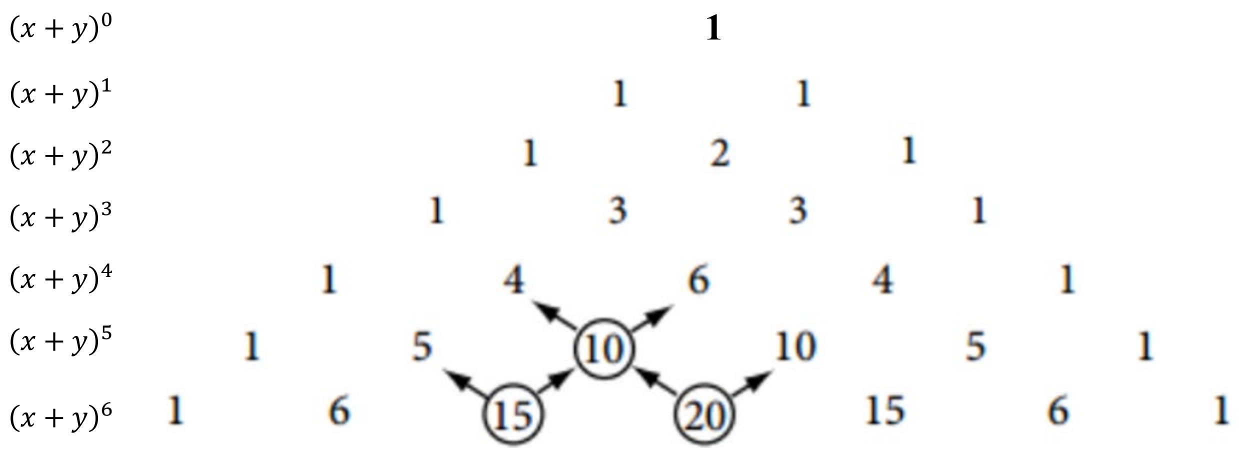
EXPANSION OF $(x + y)^n$ – PASCAL'S TRIANGLE – BINOMIAL THEOREM

By multiplying to expand the brackets, it can be shown that $(x + y)^n$ is equal to the following:

$n = 0$	$(x + y)^0 = 1$
$n = 1$	$(x + y)^1 = x + y$
$n = 2$	$(x + y)^2 = x^2 + 2xy + y^2$
$n = 3$	$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
$n = 4$	$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
$n = 5$	$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$
$n = 6$	$(x + y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

The coefficients of the$(x + y)^5$ successive powers of the expansion of $(x + y)$can be arranged in a triangular pattern called Pascal's triangle[1], as shown below.



In fact, all these coefficients can also be written as combinations, as follows:

$n = 0$ $\quad {}^0C_0$

$n = 1$ $\quad {}^1C_0 \quad {}^1C_1$

$n = 2$ $\quad {}^2C_0 \quad {}^2C_1 \quad {}^2C_2$

$n = 3$ $\quad {}^3C_0 \quad {}^3C_1 \quad {}^3C_2 \quad {}^3C_3$

$n = 4$ $\quad {}^4C_0 \quad {}^4C_1 \quad {}^4C_2 \quad {}^4C_3 \quad {}^4C_4$

$n = 5$ $\quad {}^5C_0 \quad {}^5C_1 \quad {}^5C_2 \quad {}^5C_3 \quad {}^5C_4 \quad {}^5C_5$

$n = 6$ $\quad {}^6C_0 \quad {}^6C_1 \quad {}^6C_2 \quad {}^6C_3 \quad {}^6C_4 \quad {}^6C_5 \quad {}^6C_6$

Therefore, it seems we can write the general expansion of $(x + y)^n$ as:

$$(x + y)^n = \sum_{k=0}^{n} {}^nC_k \, x^{n-k} \, y^k$$

This is known as the **binomial theorem** (to be formally demonstrated later).

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[1] after the French mathematician Blaise Pascal, although this was studied it centuries before him in India (by Pingala circa 200 AD), Persia [now Iran, by Al-Karaji (953–1029)], China [Jia Xian(1010–1070)], Germany, and Italy (https://en.wikipedia.org/wiki/Pascal%27s_triangle)

Example 32

Write the expansion of $(1+2x)^4$, using Pascal's triangle to obtain the coefficients.

Solution

Write: $(1+2x)^4 = {}^4C_0 + {}^4C_1(2x)^1 + {}^4C_2(2x)^2 + {}^4C_3(2x)^3 + {}^4C_4(2x)^4$

From Pascal's triangle: ${}^4C_0 = 1, {}^4C_1 = 4, {}^4C_2 = 6, {}^4C_3 = 4, {}^4C_4 = 1$

$$\therefore \quad (1+2x)^4 = 1 + 4\times 2x + 6\times 4x^2 + 4\times 8x^3 + 1\times 16x^4$$
$$= 1 + 8x + 24x^2 + 32x^3 + 16x^4$$

Example 33

Write the expansion of $(1-x)^3$.

Solution

Write: $(1-x)^3 = {}^3C_0 + {}^3C_1(-x)^1 + {}^3C_2(-x)^2 + {}^3C_3(-x)^3$

From Pascal's triangle: ${}^3C_0 = 1, {}^3C_1 = 3, {}^3C_2 = 3, {}^3C_3 = 1$

$$\therefore \quad (1-x)^3 = 1 + 3\times(-x) + 3\times x^2 + 1\times(-x^3)$$
$$= 1 - 3x + 3x^2 - x^3$$

Example 35

Find the coefficient of a^4 in $(1+5a)^5$.

Solution

For the expansion, you can write $T_1 = {}^5C_0$, $T_2 = {}^5C_1(5a)^1$ and so on, leading to the general term $T_{r+1} = {}^5C_r(5a)^r$.

For the term in a^4, $r = 4$: $T_5 = {}^5C_4(5a)^4 = 5\times 5^4a^4 = 3125a^4$

Hence the coefficient of a^4 is 3125.

Example 36

Use the expansion of $\left(1-\frac{1}{x}\right)^5$ to find an approximation for $\left(\frac{99}{100}\right)^5$ correct to four decimal places.

Solution

This can be written as an expansion: $\left(\frac{99}{100}\right)^5 = \left(1-\frac{1}{100}\right)^5$

$$= 1 - 5\times\frac{1}{100} + 10\times\frac{1}{100^2} - 10\times\frac{1}{100^3} + \ldots$$
$$= 1 - \frac{5}{100} + \frac{10}{10000} - \frac{10}{1000000} + \ldots$$
$$= 1 - 0.05 + 0.001 - 0.00001 + \ldots$$
$$\approx 0.9510$$

As the accuracy is only to four decimal places, the terms smaller than 0.000 01 are insignificant and can be ignored.

$\therefore \left(\frac{99}{100}\right)^5 \approx 0.9510$ correct to four decimal places.

PROOF BY INDUCTION OF THE BINOMIAL THEOREM:

For any positive integer n : $\quad (x+y)^n = \sum_{k=0}^{n} {}^{n}C_k \, x^{n-k} \, y^k \quad$ where: $\quad {}^{n}C_k = \dfrac{n!}{(n-k)! \; k!}$

Step 1: We demonstrate it is true for $n = 1$

For $n = 1$, $(x+y)^1 = x + y$ $\qquad$ whereas: $(x+y)^1 = \sum_{k=0}^{1} {}^{1}C_k \, x^{1-k} y^k$

$$(x+y)^1 = \frac{1!}{(1-0)! \; 0!} x^{1-0} \, y^0 + \frac{1!}{(1-1)! \; 1!} x^{1-1} \, y^1$$

$$(x+y)^1 = \frac{1!}{1! \; 0!} x \times 1 + \frac{1!}{0! \; 1!} x^0 \, y^1 = \frac{1}{1 \times 1} x + \frac{1}{1 \times 1} 1 \times \, y = x + y$$

Therefore $(x+y)^n = \sum_{k=0}^{n} {}^{n}C_k \, x^{n-k} \, y^k$ is indeed true for $n = 1$

Step 2: We demonstrate that if it is true for $(n-1)$, then it is true for n.

Suppose it is true for $(n-1)$: $\qquad (x+y)^{n-1} = \sum_{k=0}^{n-1} {}^{n-1}C_k \, x^{(n-1)-k} \, y^k$

$$(x+y)^n = (x+y)(x+y)^{n-1} = (x+y)\left[\sum_{k=0}^{n-1} {}^{n-1}C_k \, x^{(n-1)-k} \, y^k\right]$$

$$(x+y)^n = x\left[\sum_{k=0}^{n-1} {}^{n-1}C_k \, x^{(n-1)-k} \, y^k\right] + y\left[\sum_{k=0}^{n-1} {}^{n-1}C_k \, x^{(n-1)-k} \, y^k\right]$$

$$(x+y)^n = \sum_{k=0}^{n-1} {}^{n-1}C_k \, x^{n-k} \, y^k + \sum_{k=0}^{n-1} {}^{n-1}C_k \, x^{(n-1)-k} \, y^{k+1}$$

$$(x+y)^n = \sum_{k=0}^{n-1} {}^{n-1}C_k \, x^{n-k} \, y^k + \sum_{j=0}^{n-1} {}^{n-1}C_j \, x^{(n-1)-j} \, y^{j+1}$$

we just replaced the letter k by the letter j in the second expression

$$(x+y)^n = \sum_{k=0}^{n-1} {}^{n-1}C_k \, x^{n-k} \, y^k + \sum_{j=0}^{n-1} {}^{n-1}C_{j+1-1} \, x^{n-(j+1)} \, y^{j+1}$$

Now in the 2nd term, we substitute $j + 1 = k$ and therefore, when $j = 0$, then $k = 1$, and when $j = n - 1$, then $k = n$

$$(x+y)^n = \sum_{k=0}^{n-1} {}^{n-1}C_k \, x^{n-k} \, y^k + \sum_{k=1}^{n} {}^{n-1}C_{k-1} \, x^{n-k} \, y^k$$

$$(x+y)^n = \left[{}^{n-1}C_0 \, x^n \, y^0 + \sum_{k=1}^{n-1} {}^{n-1}C_k \, x^{n-k} \, y^k\right] + \left[\sum_{k=1}^{n-1} {}^{n-1}C_{k-1} \, x^{n-k} \, y^k + {}^{n-1}C_{n-1} \, x^{n-n} \, y^n\right]$$

$$(x+y)^n = \left[\frac{(n-1)!}{0!\,(n-1)!}x^n + \sum_{k=1}^{n-1} {}^{n-1}C_k\, x^{n-k}\, y^k\right] + \left[\sum_{k=1}^{n-1} {}^{n-1}C_{k-1}\, x^{n-k}\, y^k + \frac{(n-1)!}{0!\,(n-1)!}y^n\right]$$

$$(x+y)^n = \left[x^n + \sum_{k=1}^{n-1} {}^{n-1}C_k\, x^{n-k}\, y^k\right] + \left[\sum_{k=1}^{n-1} {}^{n-1}C_{k-1}\, x^{n-k}\, y^k + y^n\right]$$

$$(x+y)^n = x^n + \sum_{k=1}^{n-1} {}^{n-1}C_k\, x^{n-k}\, y^k + \sum_{k=1}^{n-1} {}^{n-1}C_{k-1}\, x^{n-k}\, y^k + y^n$$

$$(x+y)^n = x^n + \sum_{k=1}^{n-1}\left[{}^{n-1}C_k + {}^{n-1}C_{k-1}\right] x^{n-k}\, y^k + y^n$$

Now it can be demonstrated (see footnote[i]) that ${}^{n-1}C_k + {}^{n-1}C_{k-1} = {}^{n}C_k$

$$(x+y)^n = \frac{n!}{0!\,n!}x^n + \sum_{k=1}^{n-1} \frac{n!}{k!\,(n-k)!}x^{n-k}y^k + \frac{n!}{n!\,0!}y^n$$

Therefore: $$\boldsymbol{(x+y)^n = \sum_{k=0}^{n} {}^{n}C_k\, x^{n-k}\, y^k}$$

So if $(x+y)^n = \sum_{k=0}^{n} {}^{n}C_k\, x^{n-k}\, y^k$ is true for $(n-1)$, then it is true for n.

Step 3:

- At Step 1, we proved that $(x+y)^n = \sum_{k=0}^{n} {}^{n}C_k\, x^{n-k}\, y^k$ is true for $n = 1$
- At Step 2, we proved that if $(x+y)^n = \sum_{k=0}^{n} {}^{n}C_k\, x^{n-k}\, y^k$ is true for $(n-1)$ then it is true for n
- Therefore, by mathematical induction, it must be true for $n = 2$, and then for $n = 3$, for $n = 4$, etc., i.e. <u>it is true for any positive integer n</u>

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[i] $${}^{n-1}C_k + {}^{n-1}C_{k-1} = \frac{(n-1)!}{(n-1-k)!k!} + \frac{(n-1)!}{[n-1-(k-1)]!\,(k-1)!} = \frac{(n-1)!}{(n-1-k)!\,k!} + \frac{(n-1)!}{(n-k)!\,(k-1)!}$$

$${}^{n-1}C_k + {}^{n-1}C_{k-1} = \frac{(n-k)\,(n-1)!}{(n-k)\,(n-1-k)!\,k!} + \frac{k\,(n-1)!}{k\,(n-k)!\,(k-1)!} = \frac{(n-k)\,(n-1)!}{(n-k)!\,\,k!} + \frac{k\,(n-1)!}{(n-k)!\,\,k!}$$

$${}^{n-1}C_k + {}^{n-1}C_{k-1} = \frac{n\,(n-1)! - k\,(n-1)!}{(n-k)!\,\,k!} + \frac{k\,(n-1)!}{(n-k)!\,\,k!}$$ we expanded the first term

$${}^{n-1}C_k + {}^{n-1}C_{k-1} = \frac{n! - k\,(n-1)!}{(n-k)!\,\,k!} + \frac{k\,(n-1)!}{(n-k)!\,\,k!}$$

$${}^{n-1}C_k + {}^{n-1}C_{k-1} = \frac{n! - k\,(n-1)! + k\,(n-1)!}{k!\,(n-k)!}$$

$${}^{n-1}C_k + {}^{n-1}C_{k-1} = \frac{n!}{k!(n-k)!} = {}^{n}C_k$$ therefore: $$\boldsymbol{{}^{n-1}C_k + {}^{n-1}C_{k-1} = {}^{n}C_k}$$