VELOCITY AND ACCELERATION AS FUNCTIONS OF x

1 A particle moves in a straight line so that at time t its displacement from a fixed origin is x and its velocity is v. If its acceleration is given by $\ddot{x} = 4 + x$ and $v = 1$ when $x = 0$, find v when $x = 1$.

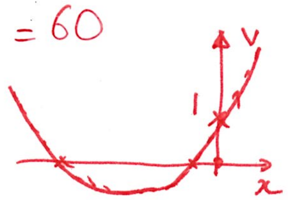
$\ddot{x} = 4 + x \iff \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = 4 + x \quad \therefore \dfrac{1}{2}v^2 = 4x + \dfrac{x^2}{2} + C$

When $x = 0$, $v = 1$, i.e. $\dfrac{1 \times 1^2}{2} = 4 \times 0 + \dfrac{0^2}{2} + C$ so $C = \dfrac{1}{2}$

$\therefore \dfrac{1}{2}v^2 = 4x + \dfrac{x^2}{2} + \dfrac{1}{2}$ so $v^2 = x^2 + 8x + 1$

So $v = \pm\sqrt{x^2 + 8x + 1}$ $\quad \Delta = 8^2 - 4 = 60$

$x = \dfrac{-8 \pm \sqrt{60}}{2}$ two negative roots. So



So when $x > 0$, v is also positive.

$\therefore v = \sqrt{x^2 + 8x + 1}$ $\quad$ Hence $v(1) = \sqrt{1^2 + 8 \times 1 + 1} = \sqrt{10}$

3 At time t, the displacement of a particle moving in a straight line is x. If the acceleration is given by $\dfrac{d^2x}{dt^2} = 3 - 4x$ and the particle starts from rest at $x = 1$, find its velocity at any position. At what other point, if any, does the particle come to rest?

$\dfrac{d^2x}{dt^2} = 3 - 4x \Rightarrow \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = 3 - 4x$

so $\dfrac{1}{2}v^2 = 3x - \dfrac{4x^2}{2} + C$ $\quad$ so $v^2 = 6x - 4x^2 + C$

at $x = 1$, $v = 0$ so $0^2 = 6 \times 1 - 4 \times 1 + C$ so $C = -2$

$v^2 = 6x - 4x^2 - 2 = 2(3x - 2x^2 - 1) = 2(x - 1)(-2x + 1)$

$v = 0$ when $x = 1$ (which we knew)

and $x = \dfrac{1}{2}$

7 A particle moves in a straight line and its acceleration at any time is given by $\frac{d^2x}{dt^2} = \sin^2 x$. Find $\frac{dx}{dt}$ given that $\frac{dx}{dt} = 1$ when $x = 0$.

$\frac{d^2x}{dt^2} = \sin^2 x \iff \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \sin^2 x$ so $\frac{1}{2}v^2 = \int \sin^2 x \, dx$

$v^2 = \int 2\sin^2 x \, dx = \int (1 - \cos 2x) \, dx = x - \frac{\sin 2x}{2} + C$

when $x = 0$ $\frac{dx}{dt} = 1$, so $1^2 = 0^2 - \frac{\sin(2\times 0)}{0} + C$ so $C = 1$

$v^2 = x - \frac{\sin 2x}{2} + 1$

$\therefore v = \sqrt{x - \frac{\sin 2x}{2} + 1}$

10 The velocity of a particle is given by $v = 4 + x^2$ m s^{-1}. $\quad v = 4 + x^2$

(a) Find the acceleration as a function of x.

(b) If initially $x = -2$ m, what is the displacement after $\frac{\pi}{4}$ seconds?

a) $a = \frac{d^2x}{dt^2} = v\frac{dv}{dx} = (4 + x^2) \times \frac{d}{dx}(4 + x^2) = (4 + x^2) \times 2x = 2x^3 + 8x$

b) $\frac{dx}{dt} = 4 + x^2$ so $\frac{dx}{4 + x^2} = dt$

$\therefore t = \int \frac{dx}{4 + x^2} = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$

At $t = 0$, $x = -2$ so $0 = \frac{1}{2}\tan^{-1}\left(\frac{-2}{2}\right) + C$

So $C = -\frac{1}{2} \times \tan^{-1}(-1) = -\frac{1}{2} \times \left(-\frac{\pi}{4}\right) = \pi/8$

So $\left(t - \frac{\pi}{8}\right) \times 2 = \tan^{-1}\left(\frac{x}{2}\right) \Rightarrow \frac{x}{2} = \tan\left[2t - \frac{\pi}{4}\right]$

So $x = 2\tan\left[2t - \pi/4\right]$ so at $t = \frac{\pi}{4}$ $x = 2\tan\left[\frac{\pi}{4}\right] = 2$

14 If $\frac{dx}{dt}=(3-x)^2$ and $x=2$ when $t=0$, find: **(a)** x as a function of t **(b)** $\frac{d^2x}{dt^2}$ as a function of x.

a) $\frac{dx}{dt}=(3-x)^2 \Rightarrow \frac{dx}{(3-x)^2}=dt \Rightarrow \int\frac{dx}{(3-x)^2}=t$

let $u=3-x$

$\frac{du}{dx}=-1$ $du=-dx$

$t=-\int\frac{du}{u^2}=-\left[\frac{u^{-2+1}}{-2+1}\right]+C=\frac{1}{u}+C=\frac{1}{3-x}+C$

At $t=0$, $x=2$ so $\frac{1}{3-2}+C=0$ $\therefore C=-1$

$\therefore t=\frac{1}{3-x}-1$

b) $t+1=\frac{1}{3-x}$ so $3-x=\frac{1}{1+t}$

$x=3-\frac{1}{1+t}=\frac{3+3t-1}{1+t}=\frac{3t+2}{1+t}$

so $\dot{x}=\frac{3(1+t)-1\times(3t+2)}{(1+t)^2}=\frac{1}{(1+t)^2}$ and $\ddot{x}=-2(1+t)^{-3}=-\frac{2}{(1+t)^3}$

$\ddot{x}=-2(3-x)^3$

15 A particle moves in a straight line. At time t its displacement from a fixed origin is x. If $\dot{x}=x+3$: $\dot{x}=x+3$

(a) express $\ddot{x}$ in terms of x **(b)** find x when $t=1$, given that $x=-2$ when $t=0$.

a) $\dot{x}=x+3$ so $\ddot{x}=\frac{d\dot{x}}{dt}=\frac{d}{dt}(x+3)=\frac{dx}{dt}+0=\dot{x}=x+3$

$\therefore \ddot{x}=x+3$

b) $\dot{x}=x+3$ i.e. $\frac{dx}{dt}=x+3$ or $\frac{dx}{x+3}=dt$.

So $\int\frac{dx}{x+3}=t$ or $t=\ln|x+3|+C$

At $t=0$, $x=-2$, i.e. $0=\ln|-2+3|+C$

$\therefore C=0$

So $t=\ln|x+3|$

So at $t=1$, $1=\ln|x+3|$, i.e. $x+3=e$

So $x=e-3$

16 The acceleration of a body moving under gravitational attraction towards a planet varies inversely as the square of its distance from the centre of the planet. This can be written as $\frac{d^2x}{dt^2}=-\frac{k}{x^2}$ where x is the distance from the centre of the planet and k is a constant. If the body starts from rest at a distance a from the centre of the planet, show that its speed at x (before it hits the planet) is given by $\frac{dx}{dt}=\sqrt{\frac{2k(a-x)}{ax}}$.

$$a = \frac{-k}{x^2} = \frac{d^2x}{dt^2} \qquad \frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{k}{x^2}$$

$$\text{So } \frac{1}{2}v^2 = \int -\frac{k\,dx}{x^2} = -k\int\frac{dx}{x^2} = -k\int x^{-2}\,dx$$

$$\frac{1}{2}v^2 = -k\left[\frac{x^{-2+1}}{-2+1}\right] + C = \frac{k}{x} + C$$

$$\text{So } v^2 = \frac{2k}{x} + C$$

$$\text{at } x = a, \quad v = 0, \quad \text{so } 0 = \frac{2k}{a} + C \quad \text{so } C = -\frac{2k}{a}$$

$$\therefore v^2 = \frac{2k}{x} - \frac{2k}{a} = 2k\left[\frac{a-x}{ax}\right]$$

$$v = \sqrt{\frac{2k(a-x)}{ax}}$$

$$\therefore \frac{dx}{dt} = \sqrt{\frac{2k(a-x)}{ax}}$$

17 A particle is moving in a straight line with its acceleration as a function of x given by $\ddot{x} = -e^{-2x}$. It is initially at the origin and travelling with a velocity of 1 metre per second.

(a) Show that $\dot{x} = e^{-x}$. **(b)** Hence show that $x = \log_e(t+1)$.

$\ddot{x} = -e^{-2x}$

a) $\frac{d^2x}{dt^2} = -e^{-2x}$, i.e. $\frac{d}{dx}\left[\frac{1}{2}v^2\right] = -e^{-2x}$

so $\frac{1}{2}v^2 = \int -e^{-2x}\,dx = -\left[\frac{e^{-2x}}{-2}\right] + C = \frac{e^{-2x}}{2} + C$

So $v^2 = e^{-2x} + C$

At $x=0$, $v=1$, so $1^2 = e^{-2\times 0} + C$ so $C=0$

$v^2 = e^{-2x} = (e^{-x})^2$ so $v = e^{-x}$ (as v has to be positive)

$\therefore \dot{x} = e^{-x}$

b) $\dot{x} = \frac{dx}{dt} = e^{-x}$, $\therefore \frac{dx}{e^{-x}} = dt$ or $e^x dx = dt$

$\int e^x dx = t + C$ so $e^x = t + C$

At $t=0$, $x=0$, so $e^0 = 0 + C$ so $C=1$

$e^x = t+1$ so $x = \ln|t+1|$

18 A particle is moving so that $\ddot{x} = 32x^3 + 48x^2 + 16x$. Initially $x = -2$ and the velocity v is -8.

(a) Show that $v^2 = 16x^2(1+x)^2$. (b) Hence, or otherwise, show that $-4t = \int \frac{1}{x(1+x)}dx$.

(c) It can be shown that for some constant C, $\log_e\left(1+\frac{1}{x}\right) = 4t + C$. Using this equation and the initial conditions, find x as a function of t.

a) $\ddot{x} = 32x^3 + 48x^2 + 16x$ $\qquad \ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

so $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 32x^3 + 48x^2 + 16x$

$\Rightarrow \frac{1}{2}v^2 = \frac{32x^4}{4} + \frac{48x^3}{3} + \frac{16x^2}{2} + C$

So $v^2 = 16x^4 + 32x^3 + 16x^2 + C$

At $x = -2$, $v = -8$, so $(-8)^2 = 16\times(-2)^4 + 32\times(-2)^3 + 16\times(-2)^2 + C$

$C = 0$

So $v^2 = 16x^4 + 32x^3 + 16x^2$

$v^2 = 16x^2[x^2 + 2x + 1] = 16x^2(x+1)^2$

b) So $v = \pm 4x(x+1)$ but at $x = -2$, $v = -8$

so $v = -4x(x+1)$ but $v = \frac{dx}{dt}$ so $\frac{dx}{dt} = -4x(x+1)$

OR $\frac{dx}{4x(1+x)} = dt$ $\therefore -4dt = \frac{dx}{x(x-1)}$ $\therefore -4t = \int \frac{dx}{x(x+1)}$

c) $\exists C \mid \ln\left(1+\frac{1}{x}\right) = 4t + C$

At $t = 0$, $x = -2$ so $\ln\left(1-\frac{1}{2}\right) = 4\times 0 + C$

so $C = \ln\left(\frac{1}{2}\right) = -\ln 2$

So $4t - \ln 2 = \ln\left(1+\frac{1}{x}\right)$

$1 + \frac{1}{x} = e^{4t-\ln 2} \Rightarrow \frac{1}{x} = e^{4t-\ln 2} - 1 = e^{4t} \times e^{-\ln 2} - 1$

$\therefore \frac{1}{x} = \frac{e^{4t}}{2} - 1 = \frac{e^{4t}-2}{2}$

so $x = \frac{2}{e^{4t}-2}$

VELOCITY AND ACCELERATION AS FUNCTIONS OF x

20 A body falls from rest so that its velocity v metres per second after t seconds is $v = 80(1 - e^{-0.4t})$.

(a) Show that the acceleration is proportional to $(80 - v)$.

(b) Calculate the distance fallen in the first five seconds.

(c) Calculate the distance fallen when $v = 60$.

a) $V = \frac{dx}{dt} = 80(1 - e^{-0.4t})$; differentiating both sides, we get

$$\frac{dV}{dt} = \frac{d}{dt}\left[80(1-e^{-0.4t})\right] = -80 \times (-0.4)\, e^{-0.4t}$$

So $\frac{dV}{dt} = a = 32\, e^{-0.4t}$

But $\frac{V}{80} - 1 = -e^{-0.4t}$ so $e^{-0.4t} = 1 - \frac{V}{80}$

and so $a = 32\left[1 - \frac{V}{80}\right] = \frac{32}{80}\left[80 - V\right]$ indeed, a is proportional to $(80 - V)$

b) $\frac{dx}{dt} = 80(1 - e^{-0.4t})$ so $dx = 80(1 - e^{-0.4t})\,dt$

So $x = 80t - \frac{80\, e^{-0.4t}}{(-0.4)} + C = 80t + 200\, e^{-0.4t} + C$

at $t = 0$, $x = 0$, so $0 = 80 \times 0 + 200\, e^{-0.4 \times 0} + C$

so $C = -200$

$x = 80t + 200\, e^{-0.4t} - 200$

At $t = 5$ $\quad x(5) = 80 \times 5 + 200\, e^{-0.4 \times 5} - 200 \simeq 227$ m

c) when $\frac{dx}{dt} = 60 = 80(1 - e^{-0.4t_1})$ so $1 - e^{-0.4t_1} = \frac{3}{4}$ $\quad e^{-0.4t_1} = \frac{1}{4}$

$\therefore t_1 = \frac{1}{0.4} \ln 4 = 2.5 \ln 4$

$x_1 = 80 \times 2.5 \ln 4 + 200 \times \frac{1}{4} - 200 = 200 \ln 4 - 150$

$x_1 \simeq 127$ m

21 A particle is brought to top speed by an acceleration that varies linearly with the distance travelled, i.e. $\ddot{x} = kx + C$ where k and C are constants. It starts from rest with an acceleration of $3\,\text{m s}^{-2}$ and reaches top speed in a distance of 160 metres. Find:

(a) the top speed **(b)** the speed when the particle has moved 80 metres.

a) $\ddot{x} = kx + C$ so $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = kx + C$ so $\frac{1}{2}v^2 = \int (kx + C)\,dx$

So $\frac{v^2}{2} = k\int x\,dx + C\int dx = \frac{kx^2}{2} + Cx + K$

So $v^2 = kx^2 + 2Cx + 2K$

At $x = 0$, $\ddot{x} = 3$ so $C = 3$, i.e. $v^2 = kx^2 + 6x + 2K$

At $x = 0$, $v = 0$ so $0^2 = k \times 0^2 + 6 \times 0 + 2K$ so $K = 0$

$v^2 = kx^2 + 6x$

$\ddot{x} = 0$ when $x = 160$ so $k \times 160 + 3 = 0$ so $k = \frac{-3}{160}$

So $v^2 = \left(\frac{-3}{160}\right)x^2 + 6x$ so $v = \sqrt{6x + \left(\frac{-3}{160}\right)x^2}$

At $x = 160$, $v = \sqrt{6 \times 160 + \left(\frac{-3}{160}\right) \times 160^2} = \sqrt{480} = 4\sqrt{30}\ \text{ms}^{-1}$

which is the top speed.

b) At $x = 80$ $v = \sqrt{6 \times 80 + \left(\frac{-3}{160}\right) \times 80^2} = \sqrt{360} = 6\sqrt{10}\ \text{ms}^{-1}$