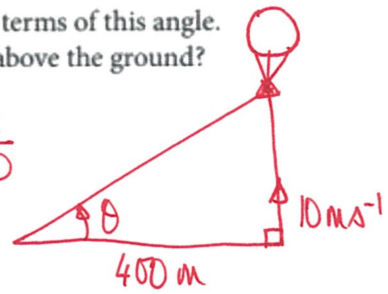
DIFFERENTIAL EQUATIONS - CHAPTER REVIEW

1 A camera at ground level is 400 metres away from a hot air balloon just prior to the balloon lifting off. The balloon lifts off and the camera records the balloon rising into the sky at a constant rate of 10 metres per second.

(a) If θ is the angle of elevation of the balloon, express the height h of the balloon in terms of this angle.

(b) How fast is the angle of elevation θ radians changing when the balloon is 300 m above the ground?



a) $h = 10t \qquad \tan\theta = \dfrac{h}{400} = \dfrac{10t}{400} = \dfrac{t}{40}$

so $h = 400\tan\theta$

b) we need $\dfrac{d\theta}{dt}$ when $h = 300$ m

$\tan\theta = \dfrac{t}{40} \quad \therefore \quad \theta = \tan^{-1}\left(\dfrac{t}{40}\right)$

$\therefore \dfrac{d\theta}{dt} = \dfrac{1/40}{1+\left(\dfrac{t}{40}\right)^2}$ But $t = \dfrac{h}{10}$ so $t = 30$ when $h = 300$ m

$\therefore$ when $h = 300$ m $\quad \dfrac{d\theta}{dt} = \dfrac{1/40}{1+\left(\dfrac{30}{40}\right)^2} = \dfrac{2}{125}$ rad s^{-1}

3 The gradient of the tangent to a curve at any point (x, y) is $\frac{x}{x+1}$, $x > -1$. If the curve passes through the point $(1, 1)$, find the equation of the curve.

$\dfrac{dy}{dx} = \dfrac{x}{x+1}$ so $dy = \dfrac{x}{x+1}\,dx$ so $\displaystyle\int dy = \int \frac{x}{x+1}\,dx$

$y = \displaystyle\int \frac{x+1-1}{x+1}\,dx = \int 1 - \frac{1}{x+1}\,dx = x - \ln|x+1| + C$

which is the general solution of the D.E.

At $x = 1$, $y = 1$ so $1 = 1 - \ln|2| + C$

so $C = \ln 2$

$y = x - \ln|x+1| + \ln 2 = x + \ln\dfrac{2}{|x+1|}$

DIFFERENTIAL EQUATIONS - CHAPTER REVIEW

7 A species of tuna is declining so that T, the number of tuna at a time t years from now, satisfies the differential equation $\frac{dT}{dt} = -0.1T$.

(a) Write the general solution to this differential equation, where $T(0) = A > 0$ is the initial population.

(b) Find the time it will take for the numbers to fall to one-quarter of their present value.

a) $\frac{dT}{dt} = -0.1T$ so $\frac{dT}{T} = -0.1\,dt$ so $\ln|T| = -0.1t + C$

$T = Ae^{-0.1t}$ general solution of the d.e.

b) when $T = A/4$ then $\frac{A}{4} = Ae^{-0.1t} \Rightarrow e^{-0.1t} = 1/4$

so $\ln\left(e^{-0.1t}\right) = \ln(1/4) \Leftrightarrow -0.1t = -\ln 4$

so $t = \frac{\ln 4}{0.1} = 10\ln 4 \simeq 13.9$ years

8 Consider the initial value problem $\frac{dy}{dx} = 2x\left(1+y^2\right)$, $y(0) = 1$. Find the exact solution to the differential equation.

$\Leftrightarrow \frac{dy}{1+y^2} = 2x\,dx$ so $\int \frac{dy}{1+y^2} = \int 2x\,dx$

So $\tan^{-1} y = x^2 + C$ so $y = \tan(x^2 + C)$ general solution of the d.e.

when $x = 0$, $y(0) = 1$ so $1 = \tan C$ so $C = \pi/4$

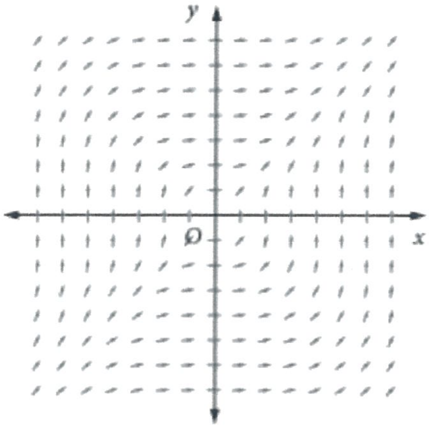
$y = \tan(x^2 + \pi/4)$ is the particular solution for which $y(0) = 1$

DIFFERENTIAL EQUATIONS - CHAPTER REVIEW

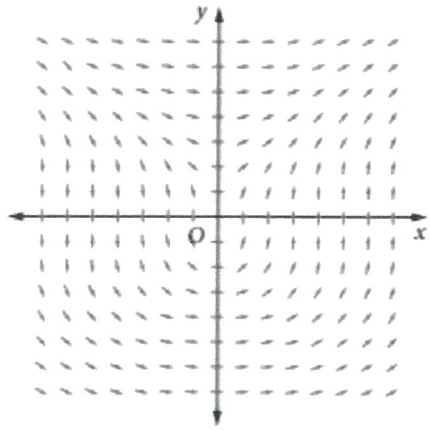
13 What is the slope field of $y' = \frac{x^2}{y^2}$?

when $x=y$, or along $y=-x$, the gradient is 1
so either A or D

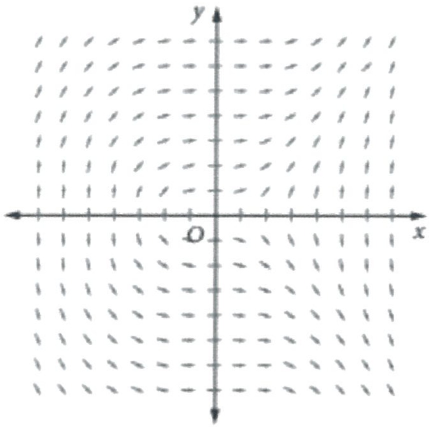
$\boxed{A}$



B

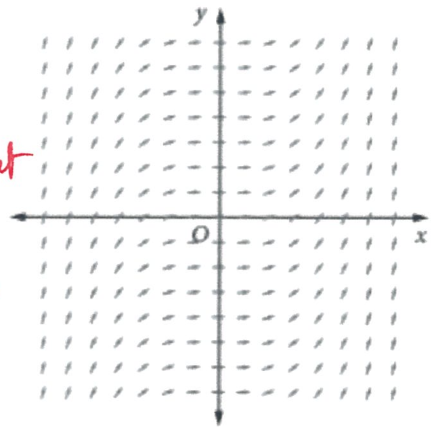


C



But not D
as the gradient
when $y=0$
has to be ∞
so $\boxed{A}$

D



16 Consider the differential equation $\frac{dy}{dx} = x - 2y$, for which the solution is $g(x)$. Which of the following statements about the particular solution that contains the point $(0, -1)$ is true at $x = 0$?

A the graph is increasing and concave up

B the graph is increasing and concave down

C the graph is decreasing and concave up

D the graph is decreasing and concave down

$\frac{dy}{dx} = x - 2y.$ at $(0,-1)$ $\frac{dy}{dx} = 0 - 2\times(-1) = 2$ which is positive.

$\therefore$ the graph is increasing at that point (response A or B)

To find the concavity, we need to calculate $\frac{d^2y}{dx^2}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[x-2y\right] = 1 - 2\frac{dy}{dx} = 1 - 2(x-2y) = 4y - 2x + 1$$

So at $(0,-1)$ $\frac{d^2y}{dx^2} = 4\times(-1) - 2\times 0 + 1 = -3$ which is negative
$\therefore$ The graph is concave down at that point

Response $\boxed{B}$

DIFFERENTIAL EQUATIONS - CHAPTER REVIEW

Consider the differential equation $\frac{dy}{dx} = y\sin x$, for which the solution is $y = f(x)$. Let $f(0) = 1$.

17 Which of the following statements about the graph of $f(x)$ are true?

(i) The slope of $f(x)$ at the point $\left(\frac{\pi}{2}, 1\right)$ is 1.
(ii) $f(x)$ has a horizontal tangent where $x = 0$.
(iii) $f(x)$ has a vertical tangent where $y = 0$.

A i only **B** ii only $\boxed{\textbf{C}}$ i and ii only **D** ii and iii only

18 The particular solution is:

A $y = e^{1 - \cos x}$ **B** $y = e^{\cos x - 1}$ **C** $y = e^{-\sin x}$ **D** $y = e^{\sin x}$

17) i) at $\left(\frac{\pi}{2}, 1\right)$ $\frac{dy}{dx} = 1 \times \sin\frac{\pi}{2} = 1$ so true

ii) when $x = 0$, $\frac{dy}{dx} = y \times \sin 0 = 0$ so true

iii) when $y = 0$ $\frac{dy}{dx} = 0 \times \sin x = 0$ so horizontal tangent (NOT vertical).

So i) and ii) are true, not iii) Response $\boxed{C}$

18) $\frac{dy}{dx} = y \sin x \iff \frac{dy}{y} = \sin x \, dx$ so $\int \frac{dy}{y} = \int \sin x \, dx$

so $\ln|y| = -\cos x + C$

so $|y| = A e^{-\cos x}$

When $x = 0$, $f(0) = 1$ so $|y| = 1 = A e^{-\cos 0} = A e^{-1}$

so $A = e$

So the particular solution is $y = e \times e^{-\cos x} = e^{1-\cos x}$

Response $\boxed{A}$

19 When added to water, 5 grams of a substance dissolves at a rate equal to 10% of the amount of undissolved chemical per hour. If x is the number of grams of *undissolved* chemical after t hours, then x satisfies the differential equation:

$\boxed{\textbf{A}}$ $\frac{dx}{dt} = -\frac{1}{10}x$ **B** $\frac{dx}{dt} = -\frac{1}{5}x$ **C** $\frac{dx}{dt} = \frac{1}{5}(10-x)$ **D** $\frac{dx}{dt} = \frac{1}{10}(5-x)$

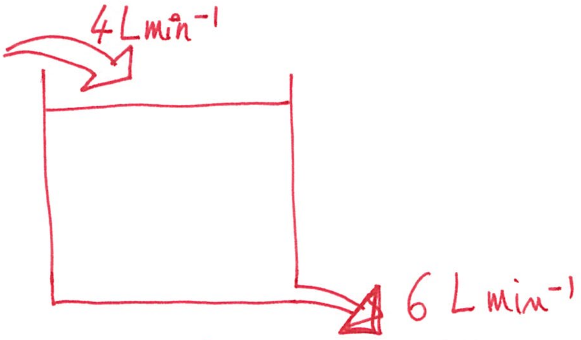
Amount of undissolved chemical x

$$\frac{dx}{dt} = -\frac{1}{10}x$$

$\boxed{A}$ **21** A quantity of sugar is dissolved in a tank containing 100 litres of pure water. At time $t = 0$ minutes, pure water is poured into the tank at a rate of 4 litres per minute. The tank is kept well stirred at all times. At the same time, the sugar solution is drained from a tap at the bottom of the tank at a rate of 6 litres per minute. A differential equation for the mass m grams of sugar in the tank is:

A $\frac{dm}{dt} = -6m$ **B** $\frac{dm}{dt} = 4 - \frac{3m}{50}$ **C** $\frac{dm}{dt} = -\frac{3m}{50-t}$ **D** $\frac{dm}{dt} = 4 - \frac{3m}{50-t}$

At $t=0$ m in water.
100 L water.



As more water is flowing out of the tank than what comes in, the tank will be empty after $\frac{100}{2} = 50$ min.

So for $t < 50$, $\frac{dm}{dt} = \underbrace{\text{rate of inflow}}_{=0} - \text{rate of outflow}$

$$\frac{dm}{dt} = -6 \times \frac{m}{V} = -6 \times \frac{m}{100-(6-4)t} = \frac{-6m}{100-2t} = \frac{-3m}{50-t}$$

Response $\boxed{C}$

22 According to Fourier's law of heat conduction, the rate of heat transfer $\frac{dQ}{dt}$ through an ice sheet in Antarctica is given by the differential equation $\frac{dQ}{dt} = \frac{k(T_w - T_a)}{h}$, where k is the thermal conductivity of the ice, h is the thickness of the ice sheet and T_w and T_a are the temperatures at the ice/water boundary and the ice/air boundary respectively.

As the water loses Q joules of heat through the ice sheet, the rate of increase in ice thickness h is given by $\frac{dh}{dQ} = \frac{1}{L\rho}$, where L is the latent heat of sea water (in other words, the amount of heat loss required to freeze 1 kilogram of it) and ρ is the density of the ice.

(a) Find the rate of increase of the ice sheet thickness $\frac{dh}{dt}$.

(b) If $h(0) = h_0$, find $h(t)$, assuming that $\frac{k(T_w - T_a)}{L\rho}$ is a positive constant.

a) $\frac{dQ}{dt} = \frac{k(T_w - T_a)}{h}$ and $\frac{dh}{dQ} = \frac{1}{L\rho}$

So $\frac{dh}{dt} = \frac{dh}{dQ} \times \frac{dQ}{dt} = \frac{1}{L\rho} \times \frac{k(T_w - T_a)}{h} = \frac{k(T_w - T_a)}{L\rho h}$

b) From the differential equation (d.e.) $h\,dh = \frac{k(T_w - T_a)}{L\rho}\,dt$

So $\int h\,dh = \int \frac{k(T_w - T_a)}{L\rho}\,dt = \frac{k(T_w - T_a)}{L\rho} \int dt$

So $\frac{h^2(t)}{2} = \frac{k(T_w - T_a)}{L\rho}t + C$

$h^2(t) = \frac{2k(T_w - T_a)t}{L\rho} + K$

at $t = 0$ $\quad h(0) = h_0$ so $K = h_0^2$

$h^2(t) = h_0^2 + \frac{2k(T_w - T_a)}{L\rho}$

$\therefore \quad h(t) = \sqrt{h_0^2 + \frac{2k(T_w - T_a)}{L\rho}}$

28 Bob's credit card bill B is initially \$15 000 and he pays 18% interest on this debt per year, compounded continuously. He decides to pay it off by transferring money from his savings account continuously at the rate of \$300 per month.

(a) Find and solve a differential equation to model the credit card balance B after t years.
(b) How much time will it take to pay off the credit card bill (to the nearest day)?
(c) What is the sum total of Bob's repayments?

Assume Bob has \$40 000 in a savings account that accumulates interest at an annual rate of 6%, also compounded continuously.

(d) Find and solve a differential equation to model the balance S of Bob's savings account.
(e) How much money will Bob have in his savings account when the debt is finally paid off (assuming no other transactions)?

a) $\frac{dB}{dt} = 0.18B - 12 \times 300 = 0.18B - 3600 = 0.18(B - 20{,}000)$

so $\frac{dB}{B - 20{,}000} = 0.18dt$ $\quad \ln|B - 20{,}000| = 0.18t + C$

$B - 20{,}000 = Ae^{0.18t}$ so $B = 20{,}000 + Ae^{9t/50}$

At $t=0$ $B = 15{,}000$ so $B = 20{,}000 + A = 15{,}000$ so $A = -5{,}000$

$B = 20{,}000 - 5{,}000\, e^{9t/50}$

b) $B=0$ when $e^{9t/50} = \frac{20{,}000}{5{,}000} = 4$ so $\frac{9t}{50} = \ln 4$ so $t = \frac{50\ln 4}{9}$

$t \approx 7.702$ years so 7 years 256 days.

c) $7.702 \times 300 \times 12 = \$27{,}726$

d) $\frac{dS}{dt} = S \times 0.06 - 3600$ so $\frac{dS}{dt} = 0.06(S - 60{,}000)$

$\frac{dS}{S - 60{,}000} = 0.06\,dt$ so $\ln(S - 60{,}000) = 0.06t + C$

so $S - 60{,}000 = Ae^{0.06t}$

when $t=0$, $S = 40{,}000$ so $-20{,}000 = Ae^{0.06\times 0} = A$.

$\therefore$ $S = 60{,}000 - 20{,}000\, e^{0.06t} = 20{,}000\left[3 - e^{3t/50}\right]$

e) when $t \approx 7.702$, i.e. $e^{9t/50} = 4$ or $(e^{3t/50})^3 = 4$ so $e^{3t/50} = \sqrt[3]{4}$

then $S = 20{,}000\left[3 - \sqrt[3]{4}\right] \approx \$28{,}251.98$

29 An abandoned open-cut mine just outside a large city has been purchased as a landfill for solid waste by a city council. When purchased, the open-cut mine had a volume of 1 million cubic metres. At the beginning of 2015, the landfill already had 100 000 cubic metres of solid waste. The volume of solid waste W in the landfill (measured in units of 100 000 cubic metres) t years after the beginning of 2015 is modelled by the solution of the differential equation $\frac{dW}{dt}=\frac{1}{10}(10-W)$, $W(0)=1$.

(a) Find the volume of solid waste in the landfill t years after 2015.

(b) Hence determine the volume of solid waste in the landfill at the beginning of 2035. (Express your answer in cubic metres, correct to the nearest cubic metre.)

a) $\frac{dW}{dt}=\frac{1}{10}(10-W)$ and $W(0)=1$

$\frac{dW}{10-W}=\frac{dt}{10}$ $\quad \int\frac{dW}{-10+W}=\frac{-1}{10}\int dt$ so $\ln|W-10|=\frac{-1}{10}t+C$

$W-10=Ae^{-t/10}$ $\quad$ But $W(0)=1$ so $1-10=A$

so $A=-9$

$W=10-9e^{-t/10}$

b) At $t=20$ (i.e in 2035)

$W=10-9e^{-20/10}=10-9e^{-2}=10-\frac{9}{e^2}\approx 8.78198$

So Volume of solid waste in $2035=878{,}198\ m^3$

30 The population $P(t)$ of penguins on an island in the Southern Ocean t years after the beginning of 2015 grows at a rate directly proportional to $1000 - P(t)$, where the constant of proportionality is k.

(a) If the population at the beginning of 2015 is 200, express the penguin population t years after the beginning of 2015 in terms of t and k.

(b) If the population after 2 years is 300, find k.

(c) Hence determine the long-term population of penguins on the island.

a) $\frac{dP}{dt} = k(1000 - P)$ so $\frac{dP}{P - 1000} = -k\,dt$

$\int \frac{dP}{P-1000} = \int -k\,dt$ so $\ln|P - 1000| = -kt + C$

so $|P - 1000| = e^{-kt+C} = A e^{-kt}$ with $A > 0$

if $|P - 1000| = P - 1000$ then $P - 1000 = A e^{-kt}$ so $P = 1000 + A e^{-kt}$ with $A > 0$

if $|P - 1000| = 1000 - P$ then $1000 - P = A e^{-kt}$ so $P = 1000 - A e^{-kt}$ with $A > 0$

These two general solutions can be combined as one, $P = 1000 + K e^{-kt}$ with K any value.

In 2015, $t = 0$, $P = 200$ so $200 = 1{,}000 + K$ so $K = -800$

So $P = 1000 - 800 e^{-kt}$

b) At $t = 2$, $P = 300$ so $300 = 1000 - 800 e^{-2k}$

so $e^{-2k} = \frac{-700}{-800} = \frac{7}{8}$ $\Rightarrow$ $-2k = \ln(7/8)$ $\therefore k = -\frac{1}{2}\ln(7/8)$

$k = \frac{1}{2}\ln(8/7)$

c) $P = 1000 - 800 e^{-\frac{t}{2}\ln 8/7} = 1000 - 800\left(e^{\ln 8/7}\right)^{-t/2} = 1000 - \left(\frac{8}{7}\right)^{-t/2} = 1000 - \left(\frac{7}{8}\right)^{t/2}$

$\lim_{t \to +\infty} P(t) = \lim_{t \to +\infty} \left(1000 - 800 \underbrace{e^{-kt}}\right)$

tends towards 0 as $k > 0$

So $\lim_{t \to +\infty} P(t) = 1000$

31 While on an unauthorised trip to the local fast food restaurant during their study period, a pair of Year 12 students are convinced that they have just seen the Prime Minister buying a hamburger. On returning to the school, their amazing discovery spreads throughout the school community at the rate $\frac{dp}{dt} = \frac{1}{10}p(1-p)$, where p is the proportion of the school community that has already heard the rumour, t minutes after their return to school.

(a) What proportion of the school community has heard the rumour when it is spreading most rapidly?
By the beginning of the afternoon period, 20% of the school community had already heard the rumour.

(b) Find $p(t)$, at time t minutes since the beginning of the afternoon period, given $\frac{1}{p(1-p)} = \frac{1}{p} - \frac{1}{p-1}$.

(c) At what time (correct to the nearest minute) is the rumour spreading most rapidly?

a) $\frac{dp}{dt} = \frac{1}{10}p(1-p)$ $\quad$ $\frac{dp}{dt}$ is the greatest when $p(1-p)$ is the greatest.

$f(p) = p(1-p) = p - p^2$ this function has a maximum (as it's a concave down parabola)

$f'(p) = 1 - 2p$ the maximum is when $f'(p) = 0$, i.e. $p = \frac{1}{2}$

So the rumour spreads most rapidly when $p = \frac{1}{2}$ (i.e. half of the population)

b) $\frac{dp}{p(1-p)} = \frac{dt}{10}$ so $\int \left[\frac{1}{p} - \frac{1}{p-1}\right] dp = \int \frac{dt}{10}$

$\therefore \int \frac{dp}{p} - \int \frac{dp}{p-1} = \frac{1}{10}\int dt \Rightarrow \ln|p| - \ln|p-1| = \frac{t}{10} + C$

so $\ln\left|\frac{p(t)}{p(t)-1}\right| = \frac{t}{10} + C$ so $\frac{p}{p-1} = Ae^{t/10}$

$\Leftrightarrow p(t) = [p(t) - 1]Ae^{t/10} \Leftrightarrow p(t)[1 - Ae^{t/10}] = -Ae^{t/10}$

so $p(t) = \frac{Ae^{t/10}}{Ae^{t/10} - 1} = \frac{1}{1 - \frac{1}{Ae^{t/10}}} = \frac{1}{1 - Ke^{-t/10}}$ K constant

For $t = 0$, $p(0) = 0.2$ so $0.2 = \frac{1}{1-K}$ so $1 - K = \frac{1}{0.2} = 5$

$K = -4$

$p(t) = \frac{1}{1 + 4e^{-t/10}}$

c) $p(t) = \frac{1}{2} = \frac{1}{1 + 4e^{-t/10}}$

So $1 + 4e^{-t/10} = 2 \Leftrightarrow e^{-t/10} = \frac{1}{4} \Leftrightarrow \frac{-t}{10} = -\ln 4 \Leftrightarrow t = 10\ln 4$

$t \approx 14$ minutes.