RESISTANCE PROPORTIONAL TO SQUARE OF VELOCITY

In section 6.6 you considered projectile motion with resistance proportional to the velocity. In this section you will extend this to consider projectile motion with resistance proportional to the square of the velocity.

For air resistance that is proportional to the square of the velocity, the resistance can be represented as a vector that is opposite in direction to the velocity vector, with a magnitude equal to the magnitude of the velocity vector squared, multiplied by a constant k.

Using this definition to derive the parametric equations of the resistance in the two-dimensional case:

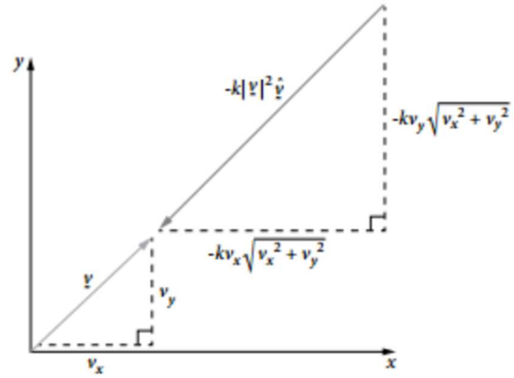
$$\underset{\sim}{R} = -k\left|\underset{\sim}{v}\right|^2 \hat{\underset{\sim}{v}} \quad \text{where } \hat{\underset{\sim}{v}} = \frac{\underset{\sim}{v}}{\left|\underset{\sim}{v}\right|} \text{ and so}$$

$$\underset{\sim}{R} = -k\left(\sqrt{v_x^{\,2} + v_y^{\,2}}\right)^2 \frac{\underset{\sim}{v}}{\sqrt{v_x^{\,2} + v_y^{\,2}}}$$

$$\underset{\sim}{R} = -k\sqrt{v_x^{\,2} + v_y^{\,2}}\,\underset{\sim}{v}$$

$$\therefore R_x = -kv_x\sqrt{v_x^{\,2} + v_y^{\,2}}$$

$$R_y = -kv_y\sqrt{v_x^{\,2} + v_y^{\,2}}$$



The horizontal and vertical components of the air resistance are given by $-kv_x\sqrt{v_x^{\,2} + v_y^{\,2}}$ and $-kv_y\sqrt{v_x^{\,2} + v_y^{\,2}}$, respectively. These components are not independent of each other, which means there is no straightforward way to find the position vectors by integration. Instead, projectile motion can be examined by using a mathematical model where the components of the resistance are simplified and approximated to be simply proportional to the velocity vector components squared. That is, $-kv_x\sqrt{v_x^{\,2} + v_y^{\,2}}$ is approximated as $-k(v_x)^2$ and $-kv_y\sqrt{v_x^{\,2} + v_y^{\,2}}$ is approximated as $-k(v_y)^2$.

A mathematical model for air resistance with the square of the velocity

A particle of mass m is launched at time $t = 0$, from ground level on a flat plane, at an angle of θ to the horizontal with an initial velocity of u m s^{-1}. In addition to gravity, there is an air resistance force that acts in the opposite direction to the instantaneous direction of motion. The magnitude of this resistance force is directly proportional to the square of its instantaneous speed.

Use the standard Cartesian coordinates with the x-axis horizontal and the y-axis vertical.

Let the components of the acceleration be $\ddot{x}$ and $\ddot{y}$, the components of the velocity be $\dot{x}$ and $\dot{y}$, and the components of the displacement be x and y.

Initially, $x = 0$, $\dot{x} = u_x = u\cos\theta$, $y = 0$, $\dot{y} = u_y = u\sin\theta$.

Now the model is $m\ddot{x} = -mk(\dot{x})^2$ and $m\ddot{y} = -mg - mk(\dot{y})^2$, where k is the constant of proportionality of the resistance.

Dividing by m reduces these equations to: $\ddot{x} = -k(\dot{x})^2$, $\ddot{y} = -g - k(\dot{y})^2$.

Consider the horizontal motion and let $v_x = \dot{x}$ so that $\ddot{x} = \dfrac{dv_x}{dt}$: $\dfrac{dv_x}{dt} = -kv_x^{\,2}$

$$\frac{dv_x}{v_x^{\,2}} = -k\,dt$$

Integrate both sides: $\displaystyle\int_{u_x}^{v_x} \frac{dv_x}{v_x^{\,2}} = -k\int_0^t dt$

$$\left[\frac{-1}{v_x}\right]_{u_x}^{v_x} = -kt$$

$$\frac{-1}{v_x} + \frac{1}{u_x} = -kt$$

$$\frac{1}{v_x} = \frac{1 + ku_x t}{u_x}$$

$$v_x = \frac{u_x}{1 + ku_x t}$$

Now $v_x = \dot{x}$: $\dfrac{dx}{dt} = \dfrac{u_x}{1 + ku_x t}$

Integrate with respect to t: $x = u_x \displaystyle\int_0^t \frac{dt}{1 + ku_x t}$

$$x = \frac{1}{k}\left[\log_e\left(1 + ku_x t\right)\right]_0^t$$

$$x = \frac{1}{k}\log_e\left(1 + ku_x t\right)$$

Consider the vertical motion and let $v_y = \dot{y}$ so that $\ddot{y} = \dfrac{dv_y}{dt}$:

$$\frac{dv_y}{dt} = -g - k(v_y)^2$$

$$\frac{dv_y}{g + k(v_y)^2} = -dt$$

Integrate both sides: $\displaystyle\int_{u_y}^{v_y} \frac{dv_y}{g + k(v_y)^2} = -\int_0^t dt$

$$\frac{1}{k}\int_{u_y}^{v_y} \frac{dv_y}{\frac{g}{k} + (v_y)^2} = -\int_0^t dt$$

$$\frac{1}{k}\left[\sqrt{\frac{k}{g}}\tan^{-1}\left(\frac{v_y}{\sqrt{\frac{g}{k}}}\right)\right]_{u_y}^{v_y} = -t$$

$$\frac{1}{\sqrt{kg}}\left(\tan^{-1}\left(\frac{\sqrt{k}v_y}{\sqrt{g}}\right) - \tan^{-1}\left(\frac{\sqrt{k}u_y}{\sqrt{g}}\right)\right) = -t$$

$$\sqrt{kg}\,t = \tan^{-1}\left(\frac{\sqrt{k}u_y}{\sqrt{g}}\right) - \tan^{-1}\left(\frac{\sqrt{k}v_y}{\sqrt{g}}\right)$$

Now $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$ and $A = \tan^{-1}\left(\dfrac{\sqrt{k}u_y}{\sqrt{g}}\right)$, $B = \tan^{-1}\left(\dfrac{\sqrt{k}v_y}{\sqrt{g}}\right)$

$$\text{So: } \tan\left(\sqrt{kg}\,t\right) = \frac{\left(\frac{\sqrt{k}u_y}{\sqrt{g}}\right) - \left(\frac{\sqrt{k}v_y}{\sqrt{g}}\right)}{1 + \left(\frac{\sqrt{k}u_y}{\sqrt{g}}\right)\left(\frac{\sqrt{k}v_y}{\sqrt{g}}\right)}$$

$$\tan\left(\sqrt{kg}\,t\right) = \frac{g}{\sqrt{g}} \times \frac{\sqrt{k}u_y - \sqrt{k}v_y}{g + \sqrt{k}u_y\sqrt{k}v_y}$$

$$\tan\left(\sqrt{kg}\,t\right)\left(g + \sqrt{k}u_y\sqrt{k}v_y\right) = \sqrt{g}\left(\sqrt{k}u_y - \sqrt{k}v_y\right)$$

$$g\tan\left(\sqrt{kg}\,t\right) + kv_y u_y \tan\left(\sqrt{kg}\,t\right) = \sqrt{gk}u_y - \sqrt{gk}v_y$$

$$\sqrt{gk}v_y + kv_y u_y \tan\left(\sqrt{kg}\,t\right) = \sqrt{gk}u_y - g\tan\left(\sqrt{kg}\,t\right)$$

$$v_y = \frac{\sqrt{gk}u_y - g\tan\left(\sqrt{kg}\,t\right)}{\sqrt{gk} + ku_y\tan\left(\sqrt{kg}\,t\right)}$$

Now $v_y = \dot{y}$ and $\frac{dy}{dt} = \frac{\sqrt{gk}u\sin\theta - g\tan\left(\sqrt{gk}t\right)}{\sqrt{gk} + ku\sin\theta\tan\left(\sqrt{gk}t\right)}$, so that $y = \int \frac{\sqrt{gk}u\sin\theta - g\tan\left(\sqrt{gk}t\right)}{\sqrt{gk} + ku\sin\theta\tan\left(\sqrt{gk}t\right)}dt$.

Let $\alpha = \sqrt{gk}$ so $\alpha^2 = gk$ or $g = \frac{\alpha^2}{k}$. Thus: $y = \int \frac{\alpha u\sin\theta - g\tan(\alpha t)}{\alpha + ku\sin\theta\tan(\alpha t)}dt$

Rewriting this integrand in terms of $\sin(\alpha t)$ and $\cos(\alpha t)$:

$$y = \int \frac{\alpha u\sin\theta - \frac{g\sin(\alpha t)}{\cos(\alpha t)}}{\alpha + \frac{ku\sin\theta\sin(\alpha t)}{\cos(\alpha t)}}dt$$

$$= \int \frac{\alpha u\sin\theta\cos(\alpha t) - g\sin(\alpha t)}{\alpha\cos(\alpha t) + ku\sin\theta\sin(\alpha t)}dt$$

Now differentiate the denominator:

$$\frac{d}{dt}(\alpha\cos(\alpha t) + ku\sin\theta\sin(\alpha t)) = -\alpha^2\sin(\alpha t) + \alpha ku\sin\theta\cos(\alpha t)$$

$$= k\left(\alpha u\sin\theta\cos(\alpha t) - \frac{\alpha^2}{k}\sin(\alpha t)\right)$$

$$= k\left(\alpha u\sin\theta\cos(\alpha t) - g\sin(\alpha t)\right)$$

Thus you can write $y = \frac{1}{k}\int \frac{f'(t)}{f(t)}dt$ where $f(t) = \alpha\cos(\alpha t) + ku\sin\theta\sin(\alpha t)$, so that $y = \log_e\left|f(t)\right| + C$.

Hence: $y = \int \frac{\alpha u\sin\theta\cos(\alpha t) - g\sin(\alpha t)}{\alpha\cos(\alpha t) + ku\sin\theta\sin(\alpha t)}dt$

$$= \frac{1}{k}\int \frac{k\left(\alpha u\sin\theta\cos(\alpha t) - g\sin(\alpha t)\right)}{\alpha\cos(\alpha t) + ku\sin\theta\sin(\alpha t)}dt$$

$$= \frac{1}{k}\log_e\left|\alpha\cos(\alpha t) + ku\sin\theta\sin(\alpha t)\right| + C$$

Rewriting this with $\alpha = \sqrt{gk}$ now gives $y = \frac{1}{k}\log_e\left|\sqrt{gk}\cos\left(\sqrt{gk}t\right) + ku\sin\theta\sin\left(\sqrt{gk}t\right)\right| + C$.

When $t = 0$, $y = 0$, so $0 = \frac{1}{k}\log_e\left|\sqrt{gk}\cos 0 + ku\sin\theta\sin 0\right| + C$ and so $C = -\frac{1}{k}\log_e\left|\sqrt{gk}\right|$.

Hence: $y = \frac{1}{k}\log_e\left|\sqrt{gk}\cos\left(\sqrt{gk}t\right) + ku\sin\theta\sin\left(\sqrt{gk}t\right)\right| - \frac{1}{k}\log_e\left|\sqrt{gk}\right|$

$$= \frac{1}{k}\log_e\left|\cos\left(\sqrt{gk}t\right) + \frac{ku\sin\theta\sin\left(\sqrt{gk}t\right)}{\sqrt{gk}}\right|$$

$$= \frac{1}{k}\log_e\left|\cos\left(\sqrt{gk}t\right) + \sqrt{\frac{k}{g}}u\sin\theta\sin\left(\sqrt{gk}t\right)\right|$$

Thus the parametric equations of the path are:

$x = \frac{1}{k}\log_e\left(1+(ku\cos\theta)t\right)$ and $y = \frac{1}{k}\log_e\left|\cos\left(\sqrt{gk}t\right)+\sqrt{\frac{k}{g}}u\sin\theta\sin\left(\sqrt{gk}t\right)\right|$

To obtain y as a function of x, first solve x for t:

$$kx = \log_e\left(1+(ku\cos\theta)t\right)$$

$$e^{kx} = 1+(ku\cos\theta)t$$

$$(ku\cos\theta)t = e^{kx}-1$$

$$t = \frac{e^{kx}-1}{ku\cos\theta}$$

So $y = \frac{1}{k}\log_e\left|\cos\left(\frac{\sqrt{gk}\left(e^{kx}-1\right)}{ku\cos\theta}\right)+\sqrt{\frac{k}{g}}u\sin\theta\sin\left(\frac{\sqrt{gk}\left(e^{kx}-1\right)}{ku\cos\theta}\right)\right|$ is the equation of the path.

Example 31

A projectile is fired from the origin O with an initial velocity u m s^{-1} at an angle θ to the horizontal in a medium whose resistance is proportional to the square of the velocity. Use $g = 10$ m s^{-2}.

The parametric equations of the motion are: $\ddot{x} = -k(\dot{x})^2$, $\ddot{y} = -10-k(\dot{y})^2$

$\dot{x} = \frac{u\cos\theta}{1+ku\cos\theta t}$, $\dot{y} = \frac{\sqrt{10k}u\sin\theta-10\tan\left(\sqrt{10k}t\right)}{\sqrt{10k}+ku\sin\theta\tan\left(\sqrt{10k}t\right)}$

$x = \frac{1}{k}\log_e\left(1+ku\cos\theta t\right)$, $y = \frac{1}{k}\log_e\left|\cos\left(\sqrt{10k}t\right)+\sqrt{\frac{k}{10}}u\sin\theta\sin\left(\sqrt{10k}t\right)\right|$

The projectile is fired at an angle of 60° to the horizontal with an initial velocity of $10\sqrt{3}$ m s^{-1}. $k = 0.4$.

- **(a)** Find when the projectile reaches its greatest height (correct to two decimal places).
- **(b)** Find the greatest height (correct to two decimal places).
- **(c)** Find when the projectile hits the ground (correct to two decimal places).
- **(d)** Graph the path of the projectile.

Solution

(a) Greatest height when $\dot{y} = 0$:

$$\dot{y} = \frac{\sqrt{10k}u\sin\theta - 10\tan\left(\sqrt{10k}t\right)}{\sqrt{10k} + ku\sin\theta\tan\left(\sqrt{10k}t\right)}$$

$$= \frac{\sqrt{4}\times 10\sqrt{3}\times\frac{\sqrt{3}}{2} - 10\tan 2t}{\sqrt{4} + 0.4\times 10\sqrt{3}\times\frac{\sqrt{3}}{2}\tan 2t}$$

$$= \frac{30 - 10\tan 2t}{2 + 6\tan 2t}$$

$$= \frac{5(3 - \tan 2t)}{1 + 3\tan 2t}$$

Solve: $5(3 - \tan 2t) = 0$

$\tan 2t = 3$

$2t = 1.249$

$t = 0.62\text{s}$

Greatest height is at 0.62 seconds.

(b) $t = 0.62$:

$$y = \frac{10}{4}\log_e\left|\cos\left(\sqrt{4}\times 0.62\right) + \sqrt{\frac{0.4}{10}}\times 10\sqrt{3}\times\frac{\sqrt{3}}{2}\sin\left(\sqrt{4}\times 0.62\right)\right| = 2.5\log_e\left|\cos 1.24 + 3\sin 1.24\right| = 2.878 \approx 2.88\,\text{m}$$

(c) $y = 0$: $2.5\log_e\left|\cos 2t + 3\sin 2t\right| = 0$

$\cos 2t + 3\sin 2t = -1$

$\sqrt{10}\left(\frac{3}{\sqrt{10}}\sin 2t + \frac{1}{\sqrt{10}}\cos 2t\right) = 1$

$\sin(2t + \alpha) = \frac{1}{\sqrt{10}}$ where $\tan\alpha = \frac{1}{3}$

$2t + 0.3218 = 0.3218, \quad 2.82$

$2t = 2.4982$

$t = 1.25\,\text{s}$

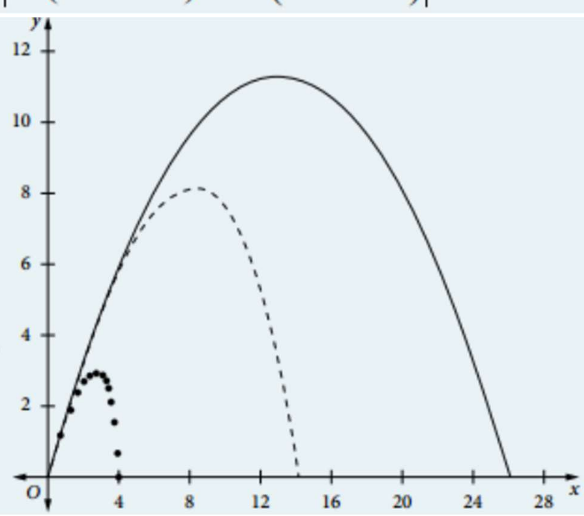
(d) The dotted line uses the parametric equations $x = 2.5\log_e\left(1 + 2\sqrt{3}t\right)$, $y = 2.5\log_e\left|\cos 2t + 3\sin 2t\right|$

or the Cartesian equation of the path $y = 2.5\log_e\left|\cos\left(\frac{2\left(e^{0.4x} - 1\right)}{2\sqrt{3}}\right) + 3\sin\left(\frac{2\left(e^{0.4x} - 1\right)}{2\sqrt{3}}\right)\right|$.

The solid line represents no resistance, the dashed line represents resistance proportional to the velocity and the dotted line represents resistance proportional to the square of the velocity.

The solid graph uses the parametric equations $x = 5\sqrt{3}t$, $y = 15t - 5t^2$ or $y = \sqrt{3}x - \frac{x^2}{15}$.

The dashed graph uses the parametric equations $x = 12.5\sqrt{3}\left(1 - e^{-0.4t}\right)$ or $y = 100\left(1 - e^{-0.4t}\right) - 25t$.



RESISTANCE PROPORTIONAL TO SQUARE OF VELOCITY

Summary of equations for projectile motion at an angle θ to the horizontal

A particle is projected from the ground with an initial velocity u at an angle θ to the horizontal where $u_x = u\cos\theta$, $u_y = u\sin\theta$. If projected from above the ground, then this initial height of projection needs to be added to the equation for y. k is the constant of proportionality for any resistance.

No resistance

$\ddot{x} = 0$	$\ddot{y} = -g$
$\dot{x} = u\cos\theta$	$\dot{y} = u\sin\theta - gt$
$x = u\cos\theta t$	$y = u\sin\theta t - \frac{1}{2}gt^2$

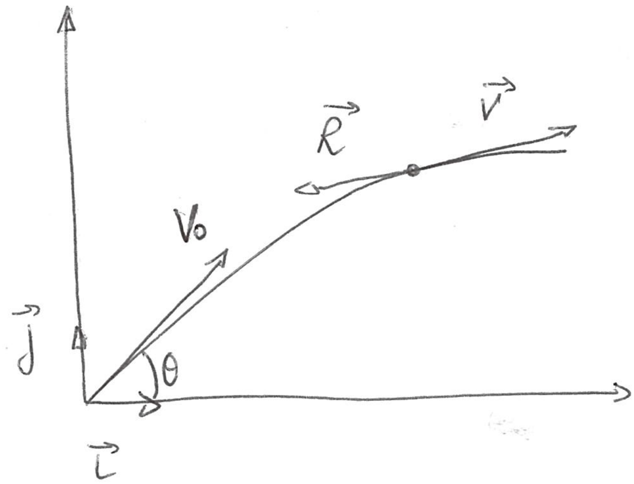
Resistance proportional to the velocity

$\ddot{x} = -k\dot{x}$	$\ddot{y} = -g - k\dot{y}$
$\dot{x} = u\cos\theta e^{-kt}$	$\dot{y} = \frac{1}{k}\left((g + ku\sin\theta)e^{-kt} - g\right)$
$x = \frac{u\cos\theta}{k}\left(1 - e^{-kt}\right)$	$y = \frac{(g + ku\sin\theta)}{k^2}\left(1 - e^{-kt}\right) - \frac{gt}{k}$

Resistance proportional to the square of the velocity—mathematical model

$\ddot{x} = -k(\dot{x})^2$	$\ddot{y} = -g - k(\dot{y})^2$
$\dot{x} = \frac{u\cos\theta}{1 + k\cos\theta t}$	$\dot{y} = \frac{\sqrt{gk}u\sin\theta - 10\tan\left(\sqrt{gk}t\right)}{\sqrt{gk} + ku\sin\theta\tan\left(\sqrt{gk}t\right)}$
$x = \frac{1}{k}\log_e\left(1 + ku\cos\theta t\right)$	$y = \frac{1}{k}\log_e\left\lvert\cos\left(\sqrt{gk}t\right) + \sqrt{\frac{k}{g}}u\sin\theta\sin\left(\sqrt{gk}t\right)\right\rvert$

$$\vec{R} = -mk\,|\vec{v}|^2 \times \hat{v}$$



$$m\vec{a} = -mg\vec{j} - mk\,|\vec{v}|^2\,\hat{v}$$

$$\vec{a} = -g\vec{j} - k\,|\vec{v}|^2\,\hat{v}$$

$$|\vec{v}|^2 = \left[\sqrt{V_x^2 + V_y^2}\right]^2 = V_x^2 + V_y^2$$

$$\left[a_x\vec{i} + a_y\vec{j}\right] = -g\vec{j} - k\left(V_x^2 + V_y^2\right) \times \left[\frac{V_x\vec{i} + V_y\vec{j}}{\sqrt{V_x^2 + V_y^2}}\right]$$

$$\Leftrightarrow \left[a_x\vec{i} + a_y\vec{j}\right] = -g\vec{j} - k\sqrt{V_x^2 + V_y^2}\,\left(V_x\vec{i} + V_y\vec{j}\right)$$

So $\begin{cases} a_x = -kV_x\sqrt{V_x^2 + V_y^2} \simeq -kV_x^2 & (1) \\ a_y = -g - kV_y\sqrt{V_x^2 + V_y^2} \simeq -g - kV_y^2 & (2) \end{cases}$ (Approximations)

$$\frac{dV_x}{dt} = -kV_x^2 \qquad \text{Equation (1)}$$

$$\frac{dV_x}{V_x^2} = -k\,dt \qquad \int_{V_0\cos\theta}^{V_x} \frac{dV_x}{V_x^2} = -k\int_0^t dt$$

$$\left[-\frac{1}{V_x}\right]_{V_0\cos\theta}^{V_x} = -kt$$

$$\left[\frac{1}{V_x}\right]_{V_0\cos\theta}^{V_x} = kt \qquad \text{so } \frac{1}{V_x} - \frac{1}{V_0\cos\theta} = kt$$

(1)

$$\frac{1}{V_x} = \frac{1}{V_0 \cos\theta} + kt = \frac{1 + kV_0\cos\theta\, t}{V_0\cos\theta}$$

$$\text{so} \quad V_x = \frac{V_0\cos\theta}{1 + [kV_0\cos\theta]_k t} = \frac{dx}{dt}$$

$$\int_0^x dx = \int_0^t \frac{V_0\cos\theta}{1+(kV_0\cos\theta)t}\, dt = V_0\cos\theta \int_0^t \frac{dt}{1+(kV_0\cos\theta)\, t}$$

$$x(t) = V_0\cos\theta \left[\frac{\ln[1+(kV_0\cos\theta)t]}{kV_0\cos\theta} \right]_0^t$$

$$x(t) = \frac{1}{k}\left[\ln[1+(kV_0\cos\theta)t] - \underbrace{\ln[1+(kV_0\cos\theta)\times 0]}_{=0} \right]$$

$$\text{So} \quad x(t) = \frac{1}{k}\ln\left[1+(kV_0\cos\theta)t\right] \qquad \text{Equation } Ⓐ$$

From ②

$$\frac{dV_y}{dt} = -g - kV_y^2$$

$$\frac{dV_y}{g + kV_y^2} = -dt$$

$$\int_{V_0\sin\theta}^{V_y} \frac{dV_y}{g + kV_y^2} = -\int_0^t dt$$

$$\frac{1}{k}\int_{V_0\sin\theta}^{V_y} \frac{dV_y}{(g/k) + V_y^2} = -t$$

$$\frac{1}{k}\left[\frac{1}{\sqrt{g/k}}\tan^{-1}\left(\frac{V_y}{\sqrt{g/k}}\right)\right]_{V_0\sin\theta}^{V_y} = -t$$

$$\frac{1}{k}\frac{\sqrt{k}}{\sqrt{g}}\left[\tan^{-1}\left(\sqrt{\frac{k}{g}}\times V_y\right)\right]_{V_0\sin\theta}^{V_y} = -t$$

$$\tan^{-1}\left(\sqrt{\frac{k}{g}}\times V_y\right) - \tan^{-1}\left(\sqrt{\frac{k}{g}}\times V_0\sin\theta\right) = -\sqrt{kg}\times t$$

So $$\sqrt{kg}\,t = \tan^{-1}\left(\sqrt{\frac{k}{g}}\,\cancel{V}_0\sin\theta\right) - \tan^{-1}\left(\sqrt{\frac{k}{g}}\,V_y\right)$$

$$\tan\left(\sqrt{kg}\,t\right) = \tan\left[\tan^{-1}\left(\sqrt{\frac{k}{g}}\,V_0\sin\theta\right) - \tan^{-1}\left(\sqrt{\frac{k}{g}}\,V_y\right)\right]$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B}$$, hence:

$$\tan\left(\sqrt{kg}\,t\right) = \frac{\sqrt{k/g}\,V_0\sin\theta - \sqrt{k/g}\,V_y}{1+\sqrt{\frac{k}{g}}\,V_0\sin\theta\times\sqrt{\frac{k}{g}}\,V_y}$$

$$\tan\left(\sqrt{kg}\,t\right) = \frac{\left(V_0\sin\theta - V_y\right)\frac{\sqrt{k}}{\sqrt{g}}}{1+\frac{k}{g}\,V_0\sin\theta\times V_y}$$

$$\tan\left(\sqrt{kg}\,t\right) = \frac{\sqrt{gk}\,\left(V_0\sin\theta - V_y\right)}{g + k\,V_0\sin\theta\,V_y}$$

$$\left[g+kV_0\sin\theta\,V_y\right]\tan\left(\sqrt{kg}\,t\right) = \sqrt{gk}\left(V_0\sin\theta - V_y\right)$$

$$V_y\left[kV_0\sin\theta\tan\left(\sqrt{kg}\,t\right)+\sqrt{gk}\right] = \sqrt{gk}\,V_0\sin\theta - g\tan\left(\sqrt{kg}\,t\right)$$

So $V_y = \dfrac{\sqrt{gk}\; V_0 \sin\theta - g \tan(\sqrt{kg}\, t)}{k V_0 \sin\theta \tan(\sqrt{kg}\, t) + \sqrt{gk}} = \dfrac{dy}{dt}$

$$\frac{dy}{dt} = \frac{\sqrt{gk}\; V_0 \sin\theta - g\, \dfrac{\sin(\sqrt{kg}\, t)}{\cos(\sqrt{kg}\, t)}}{k V_0 \sin\theta \times \dfrac{\sin(\sqrt{kg}\, t)}{\cos(\sqrt{kg}\, t)} + \sqrt{gk}}$$

$$\frac{dy}{dt} = \frac{\sqrt{gk}\; V_0 \sin\theta \cos[\sqrt{kg}\, t] - g \sin[\sqrt{kg}\, t]}{k V_0 \sin\theta \times \sin[\sqrt{kg}\, t] + \sqrt{gk} \cos[\sqrt{kg}\, t]}$$

let $f(t) = k V_0 \sin\theta \times \sin[\sqrt{kg}\, t] + \sqrt{gk} \cos[\sqrt{kg}\, t]$ $\circledast$

then $\dfrac{df}{dt} = k V_0 \sin\theta \times \sqrt{kg} \cos[\sqrt{kg}\, t] + \sqrt{gk}\sqrt{kg}\,(-\sin\sqrt{kg}\, t)$

So $\dfrac{df}{dt} = k V_0 \sin\theta \sqrt{kg} \cos[\sqrt{kg}\, t] - gk \sin[\sqrt{kg}\, t]$

$$f'(t) = k \Big[\underbrace{\sqrt{kg}\; V_0 \sin\theta \cos[\sqrt{kg}\, t] - g \sin[\sqrt{kg}\, t]}_{\text{which is the numerator}} \Big]$$

Hence, $\dfrac{dy}{dt} = \dfrac{\frac{1}{k} f'(t)}{f(t)} = \dfrac{1}{k} \dfrac{f'(t)}{f(t)}$

So $dy = \dfrac{1}{k} \dfrac{f'(t)}{f(t)} \times dt$

$$\int_0^y dy = \frac{1}{k} \int_0^t \frac{f'(t)}{f(t)}\, dt = \frac{1}{k} \Big[\ln |f(t)| \Big]_0^t$$

$$y = \frac{1}{k} \ln \left| \frac{f(t)}{f(0)} \right|$$

$$f(0) = \underbrace{k V_0 \sin\theta \sin\left[\sqrt{kg} \times 0\right]}_{=0} + \underbrace{\sqrt{gk} \cos\left[\sqrt{gk} \times 0\right]}_{= \sqrt{gk}}$$

$$\therefore y = \frac{1}{k} \ln \left| \frac{k V_0 \sin\theta \sin\left[\sqrt{kg}\, t\right] + \sqrt{gk} \cos\left[\sqrt{kg}\, t\right]}{\sqrt{gk}} \right|$$

$$y(t) = \frac{1}{k} \ln \left[\sqrt{\frac{k}{g}} V_0 \sin\theta \sin\left(\sqrt{kg}\, t\right) + \cos\left(\sqrt{kg}\, t\right) \right]$$

Equation B

Thus the parametric equations of the motion are:

A $$x(t) = \frac{1}{k} \ln \left[1 + (k V_0 \cos\theta)\, t \right]$$

B $$y(t) = \frac{1}{k} \ln \left[\sqrt{\frac{k}{g}} V_0 \sin\theta \sin\left(\sqrt{kg}\, t\right) + \cos\left(\sqrt{kg}\, t\right) \right]$$

From A, we get: $kx = \ln\left[1 + (k V_0 \cos\theta) t\right]$

so $1 + (k V_0 \cos\theta) t = e^{kx}$

$$(k V_0 \cos\theta)\, t = e^{kx} - 1$$

$$t = \frac{1}{k V_0 \cos\theta} \left[e^{kx} - 1 \right]$$

which we substitute into B to get the Cartesian equation.

$$y = \frac{1}{k} \ln\left[\sqrt{\frac{k}{g}}\, V_0 \sin\theta \sin\left(\sqrt{kg} \times \frac{1}{k V_0 \cos\theta}\left(e^{kx}-1\right)\right)\right.$$

$$\left. + \cos\left(\sqrt{kg} \times \frac{1}{k V_0 \cos\theta}\left(e^{kx}-1\right)\right)\right]$$

which is the Cartesian equation of the path. ☺