ARITHMETIC OF COMPLEX NUMBERS

1 $i^5 = i \times i^2 \times i^2 = i \times (-1) \times (-1) = i$

A 1 B -1 $\boxed{\text{C}}$ i D $-i$

2 Solve the following equations.

(a) $z^2 + 9 = 0$ (b) $z^2 + 25 = 0$ (c) $z^2 + 2z + 17 = 0$

a) $z^2 = -9 = (3i)^2$ so $z = 3i$ or $z = -3i$

b) $z^2 = -25 = (5i)^2$ so $z = 5i$ or $z = -5i$

c) $\Delta = 2^2 - 4 \times 17 = -64 = (8i)^2$ so $z = \dfrac{-2 \pm 8i}{2}$

$z = -1 \pm 4i$

3 Simplify:

(a) i^3 (b) i^4 (c) i^6 (d) i^7 (e) i^8

a) $i^3 = i^2 \times i = -i$ b) $i^4 = i^2 \times i^2 = (-1) \times (-1) = 1$

c) $i^6 = i^2 \times i^2 \times i^2 = (-1)^3 = -1$ d) $i^7 = i^6 \times i = (-1) \times i = -i$ e) $i^8 = (i^4)^2 = 1^2 = 1$

4 If $z = 5 - 2i$, find:

a)	$\bar{z} = 5 + 2i$
b)	$z\bar{z} = (5-2i)(5+2i) = 25 - 10i + 10i - 4i^2 = 25 + 4 = 29$
c)	$z^2 = (5-2i)^2 = (5-2i)(5-2i) = 25 - 20i + (2i)^2 = 25 - 20i - 4 = 21 - 20i$
d)	$(z - \bar{z})^2 = (5 - 2i - (5+2i))^2 = (-4i)^2 = 16i^2 = -16$
e)	$\dfrac{z-1}{z-i} = \dfrac{5-2i-1}{5-2i-i} = \dfrac{4-2i}{5-3i} = \dfrac{(4-2i)(5+3i)}{(5-3i)(5+3i)}$ $= \dfrac{20+12i-10i-6i^2}{25+9} = \dfrac{20+6+2i}{34} = \dfrac{26+2i}{34} = \dfrac{13}{17} + \dfrac{i}{17}$

f) $z^{-1} = \frac{1}{5-2i} = \frac{(5+2i)}{(5-2i)(5+2i)} = \frac{5+2i}{25-(2i)^2} = \frac{5+2i}{25+4}$

$z^{-1} = \frac{5+2i}{29} = \frac{5}{29} + \frac{2i}{29}$

5 Simplify:

a) $(3+5i)+(7-2i) = 3+7+5i-2i = 10+3i$

b) $(4+7i)-(-2+9i) = 4+2+7i-9i = 6-2i$

c) $(5+2i)(3-4i) = 15-20i+6i+2i\times(-4i)$

$= 15-14i+8$

$= 23-14i$

d) $(7-3i)(7+3i) = 49-(3i)^2 = 49+9 = 58$

e) $(2-5i)^2 = 4-20i+(5i)^2$

$= 4-20i-25$

$= -21-20i$

f) $i^{17} = i^{16}\times i = (i^2)^8 \times i = (-1)^8 \times i = i$

g) $(\sqrt{3}+2i)(\sqrt{3}-2i) = 3+4 = 7$

h) $\frac{1}{2+3i} = \frac{2-3i}{(2+3i)(2-3i)} = \frac{2-3i}{4+9}$

$= \frac{2-3i}{13} = \frac{2}{13} - \frac{3}{13}i$

i) $\frac{8+5i}{4-3i} = \frac{(8+5i)(4+3i)}{(4-3i)(4+3i)} = \frac{32+20i+24i-15}{16+9}$

$= \frac{17+44i}{25} = \frac{17}{25} + \frac{44}{25}i$

j) $\frac{3i}{2+5i} + \frac{2}{2-5i} = \frac{3i(2-5i)+2(2+5i)}{(2+5i)(2-5i)}$

$= \frac{6i+15+4+10i}{4+25} = \frac{19}{29} + \frac{16}{29}i$

6 Find real numbers x and y such that:

(a) $(x+iy)(2-3i) = -13i$ **(b)** $(1+i)x + (2-3i)y = 10$

a) $2x+3y+i(2y-3x) = -13i$

So $\begin{cases} 2x+3y=0 \\ 2y-3x=-13 \end{cases}$ (equalling real and imaginary parts on both sides)

$\Leftrightarrow \begin{cases} y = -\frac{2}{3}x \\ 2\times\left(-\frac{2x}{3}\right) - 3x = -13 \quad ② \end{cases}$

$② \Rightarrow -\frac{4x}{3} - 3x = -13$

$\Rightarrow -\frac{13x}{3} = -13$

so $x = 3$

and $y = -\frac{2}{3}\times 3 = -2$

$\therefore$ $x=3$ and $y=-2$

b) $x+2y+i(x-3y) = 10$

we equal real and imaginary parts on both sides.

$\begin{cases} x+2y = 10 \\ x-3y = 0 \end{cases}$ so $x = 3y$.

and the 1st equation becomes:

$3y+2y = 10$ so $y = 2$

and $\therefore$ $x = 3\times 2 = 6$

$\therefore$ $x = 6$ and $y = 2$

ARITHMETIC OF COMPLEX NUMBERS

7 If $z_1 = 3 + i$ and $z_2 = 2 - 3i$, find:

(a) $(z_1 - z_2)^2 = [(3+i)-(2-3i)]^2 = [1+4i]^2 = 1-4^2+8i$

— $= -15+8i$

(b) $\bar{z}_1 \times \bar{z}_2 = (3-i)\times(2+3i) = 6-2i+9i-i\times(3i)$

— $= 6+3+7i = 9+7i$

(c) $\overline{z_1 z_2} = \overline{(3+i)(2-3i)} = \overline{(6+2i-9i+3)} = \overline{(9-7i)}$

— $= 9+7i$

(d) $\dfrac{z_1 - z_2}{z_1 + z_2} = \dfrac{(3+i)-(2-3i)}{(3+i)+(2-3i)} = \dfrac{3-2+i+3i}{5+i-3i} = \dfrac{1+4i}{5-2i}$

— $= \dfrac{(1+4i)(5+2i)}{(5-2i)(5+2i)} = \dfrac{5+20i+2i-8}{25+4} = \dfrac{-3}{29} + \dfrac{22i}{29}$

8 Find the linear factors of the following expressions.

(a) $z^2 + 9$ **(b)** $z^2 + 36$ **(c)** $(z-3)^2 + 16$ **(d)** $(2z+3)^2 + 8$

a) $z^2+9 = z^2-(3i)^2 = (z-3i)(z+3i)$

b) $z^2+36 = z^2-(6i)^2 = (z-6i)(z+6i)$

c) $(z-3)^2+16 = (z-3)^2-(4i)^2 = [(z-3)-4i][(z-3)+4i]$

— $= [z-3-4i][z-3+4i]$

d) $(2z+3)^2+8 = (2z+3)^2-(\sqrt{8}i)^2 = [2z+3-\sqrt{8}i][2z+3+\sqrt{8}i]$

— $= [2z+3-2\sqrt{2}i][2z+3+2\sqrt{2}i]$

(e) $z^2+2z+26$ **(f)** $z^2-6z+20$ **(g)** $2z^2+2z+4$ **(h)** z^3+1000

e) $\Delta = 4 - 4 \times 26 = -100 = (10i)^2$

so $z = \dfrac{-2 \pm 10i}{2} = -1 \pm 5i$

$\therefore\ z^2+2z+26 = [z-(-1+5i)][z-(-1-5i)] = [z+1-5i][z+1+5i]$

f) $\Delta = 6^2 - 4 \times 20 = -44 = (2\sqrt{11}\,i)^2$

So $z = \dfrac{6 \pm 2\sqrt{11}\,i}{2} = 3 \pm \sqrt{11}\,i$

$\therefore\ z^2-6z+20 = [z-(3+\sqrt{11}\,i)][z-(3-\sqrt{11}\,i)] = [z-3-\sqrt{11}\,i][z-3+\sqrt{11}\,i]$

g) $\Delta = 4 - 4 \times 4 \times 2 = -28 = (2\sqrt{7}\,i)^2$

So $z = \dfrac{-2 \pm 2\sqrt{7}\,i}{2 \times 2} = -\dfrac{1}{2} \pm \dfrac{\sqrt{7}}{2}i$

$\therefore\ 2z^2+2z+4 = 2(z^2+z+2) = 2\left[z-\left(-\frac{1}{2}+\frac{\sqrt{7}}{2}i\right)\right]\left[z-\left(-\frac{1}{2}-\frac{\sqrt{7}}{2}i\right)\right]$

$= 2\left[z+\frac{1}{2}-\frac{\sqrt{7}}{2}i\right]\left[z+\frac{1}{2}+\frac{\sqrt{7}}{2}i\right]$

h) (-10) is an obvious root, so $z^3+1000 = (z+10)(z^2-10z+100)$

Now we need to factorise the quadratic term.

$\Delta = 10^2 - 4 \times 100 = -300 = (10\sqrt{3}\,i)^2$

$z = \dfrac{10 \pm 10\sqrt{3}\,i}{2} = 5 \pm 5\sqrt{3}\,i$

$\therefore\ z^3+1000 = (z+10)[z-(5+5\sqrt{3}\,i)][z-(5-5\sqrt{3}\,i)]$

$= (z+10)[z-5-5\sqrt{3}\,i][z-5+5\sqrt{3}\,i]$

9 Solve the equation: **(a)** $2z - 1 = (4 - i)^2$ **(b)** $\frac{z-2}{z} = 1 + i$

a) $\Leftrightarrow 2z - 1 = 16 - 8i - 1$

$\Leftrightarrow 2z - 1 = 15 - 8i$

$\Leftrightarrow 2z = 16 - 8i$

$\Leftrightarrow z = 8 - 4i$

b) $\Leftrightarrow z - 2 = z(1 + i)$

$\Leftrightarrow z - 2 = z + iz$

$\Leftrightarrow -2 = iz$

$\Leftrightarrow iz = -2$

$\Leftrightarrow z = -\frac{2}{i}$

$\Leftrightarrow z = \frac{-2 \times i}{i^2}$

$\Leftrightarrow z = \frac{-2i}{(-1)}$

so $z = 2i$

12 **(a)** Show that $\sqrt{3} - i$ is a root of the equation $z^3 - (\sqrt{3} - i)z^2 + 9z - 9\sqrt{3} + 9i = 0$.

(b) Find the other two solutions of the equation.

(c) Use your answer to part **(b)** to verify that the results for the sum of roots, for the sum of products of pairs of roots and for the product of roots of a cubic equation are true when the coefficients and roots are complex numbers.

a) $(\sqrt{3}-i)^3 - (\sqrt{3}-i)(\sqrt{3}-i)^2 + 9(\sqrt{3}-i) - 9\sqrt{3} + 9i = (\sqrt{3}-i)^3 - (\sqrt{3}-i)^3 + 9\sqrt{3} - 9i - 9\sqrt{3} + 9i$

$= 0$ indeed,

$\therefore (\sqrt{3}-i)$ is a root of the equation $z^3 - (\sqrt{3}-i)z + 9z - 9\sqrt{3} + 9i = 0$

b) as $(\sqrt{3}-i)$ is a root, we can factorise by $(z - (\sqrt{3}-i))$

(i.e by $(z - \sqrt{3} + i)$)

$z^3 - (\sqrt{3}-i)z^2 + 9z - 9\sqrt{3} + 9i = z^2[z - (\sqrt{3}-i)] + 9[z - \sqrt{3} + i]$

$= [z - (\sqrt{3}-i)][z^2 + 9]$

$= [z - (\sqrt{3}-i)][z - 3i][z + 3i]$

So the two other solutions are $3i$ and $(-3i)$

c) $\alpha + \beta + \gamma = \sqrt{3} - i + 3i + (-3i) = \sqrt{3} - i$

which is equal to $\left(\frac{-b}{a}\right)$ indeed, like for cubic polynomials with real coefficients.

$\alpha\beta + \alpha\gamma + \beta\gamma = (\sqrt{3}-i)3i + (\sqrt{3}-i)(-3i) + 3i \times (-3i)$

$= 3\sqrt{3}i + 3 - 3\sqrt{3}i - 3 + 9 = 9$

which is equal to $\left(\frac{c}{a}\right)$ indeed, like for cubic polynomials with real coefficients.

$\alpha\beta\gamma = (\sqrt{3}-i) \times (3i) \times (-3i) = (\sqrt{3}-i) \times 9 = 9\sqrt{3} - 9i$

which is equal to $\left(\frac{-d}{a}\right)$ indeed, like for cubic polynomial with real coefficients.

13 Solve the following quadratic equations.

(a) $z^2 - (3-2i)z + (1-3i) = 0$

(b) $z^2 - z + (4+2i) = 0$

a) $\Delta = (3-2i)^2 - 4(1-3i)$

$\Delta = 9 - 12i - 4 - 4 + 12i$

$\Delta = 1$ two roots.

$\therefore z_1 = \frac{(3-2i)-1}{2} = 1-i$

and $z_2 = \frac{3-2i+1}{2} = 2-i$

$\Delta = 1 - 4(4+2i)$

$\Delta = -15 - 8i$

we need to find the square root of Δ i.e. the number $x+iy$ such that:

$(x+iy)^2 = -15 - 8i$

$\Leftrightarrow x^2 - y^2 + 2ixy = -15 - 8i$

$\Leftrightarrow \begin{cases} x^2 - y^2 = -15 \\ 2xy = -8 \end{cases} \Leftrightarrow \begin{cases} xy = -4 \\ x^2 - y^2 = -15 \end{cases} \Leftrightarrow \begin{cases} y = -4/x \\ x^2 - \left(\frac{-4}{x}\right)^2 = -15 \end{cases}$ equation ①

we solve equation ① $\Leftrightarrow x^4 + 15x^2 - 16 = 0$

$\Delta_2 = 15^2 - 4 \times (-16) = 289 = 17^2$

So $x^2 = \frac{-15+17}{2} = \frac{2}{2} = 1$ (the other solution is impossible as x^2 is positive, as x is a real number)

So $x = \pm 1$

For $x = 1$, $y = -4$, so $-15 - 8i = (1-4i)^2$

and $\therefore z = \frac{1 + (1-4i)}{2} = 1 - 2i$ or $z = \frac{1-(1-4i)}{2} = 2i$

For $x = -1$, $y = 4$ so $-15 - 8i = (-1+4i)^2$

and $\therefore z = \frac{1-(-1+4i)}{2} = 1 - 2i$ or $z = \frac{1+(-1+4i)}{2} = 2i$

So two solutions are $2i$ and $(1-2i)$

14 Let $z = a + ib$ where a, b are real. Prove that there are always two square roots of z except when $a = b = 0$.

We look for the number $x+iy$ such that $(x+iy)^2 = a+ib$

$\Leftrightarrow x^2 - y^2 + 2ixy = a + ib$

So $\begin{cases} x^2 - y^2 = a \\ 2xy = b \end{cases} \Leftrightarrow \begin{cases} y = \dfrac{b}{2x} \\ x^2 - \dfrac{b^2}{4x^2} = a \end{cases}$ equation ①

① $\Leftrightarrow 4x^4 - b^2 = 4ax^2 \Leftrightarrow 4x^4 - 4ax^2 - b^2 = 0$

$\Leftrightarrow x^4 - ax^2 - \dfrac{b^2}{4} = 0$ $\Delta = a^2 + b^2$ always positive

So the solution for x^2 is $\dfrac{a + \sqrt{a^2+b^2}}{2}$

(the other solution is not possible as x^2 has to be positive, as x is real)

So either $x = \sqrt{\dfrac{a+\sqrt{a^2+b^2}}{2}}$

in which case $y = \dfrac{b}{2\sqrt{\dfrac{a+\sqrt{a^2+b^2}}{2}}}$

OR $x = -\sqrt{\dfrac{a+\sqrt{a^2+b^2}}{2}}$

in which case $y = \dfrac{-b}{2\sqrt{\dfrac{a+\sqrt{a^2+b^2}}{2}}}$

So indeed, there are always two square roots of z, except when $a = b = 0$

ARITHMETIC OF COMPLEX NUMBERS

15 If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, show that the following equations are true.

(a) $z_1 + \overline{z_1} = 2 \times \mathrm{Re}(z_1)$

$z_1 + \overline{z_1} = x_1 + iy_1 + x_1 - iy_1$

$\quad = 2x_1$

$\quad = 2\,\mathrm{Re}(z_1)$

(b) $z_1 - \overline{z_1} = 2 \times \mathrm{Im}(z_1) \times i$

$z_1 - \overline{z_1} = x_1 + iy_1 - (x_1 - iy_1)$

$\quad = 2iy_1$

$\quad = 2i\,\mathrm{Im}(z_1)$

(c) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

$\overline{z_1 + z_2} = \overline{(x_1 + iy_1) + (x_2 + iy_2)}$

$\quad = \overline{x_1 + x_2 + i(y_1 + y_2)}$

$\quad = x_1 + x_2 - i(y_1 + y_2)$

$\quad = x_1 - iy_1 + x_2 - iy_2$

$\quad = \overline{z_1} + \overline{z_2}$

(d) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

$\overline{z_1 - z_2} = \overline{x_1 + iy_1 - (x_2 + iy_2)}$

$\quad = \overline{x_1 - x_2 + i(y_1 - y_2)}$

$\quad = x_1 - x_2 - i(y_1 - y_2)$

$\quad = x_1 - iy_1 - x_2 + iy_2$

$\quad = \overline{z_1} - \overline{z_2}$

(e) $\overline{z_1 \times z_2} = \overline{z_1} \times \overline{z_2}$

$\overline{z_1 \times z_2} = \overline{(x_1 + iy_1)(x_2 + iy_2)}$

$\quad = \overline{x_1x_2 - y_1y_2 + i(x_2y_1 + x_1y_2)}$

$\quad = x_1x_2 - y_1y_2 - i(x_2y_1 + x_1y_2)$

whereas $\overline{z_1} \times \overline{z_2} = (x_1 - iy_1) \times (x_2 - iy_2)$

$\quad = x_1x_2 - y_1y_2 - i(x_2y_1 + x_1y_2)$

$\therefore$ indeed, the quantities are the same, so

$\overline{z_1 \times z_2} = \overline{z_1} \times \overline{z_2}$