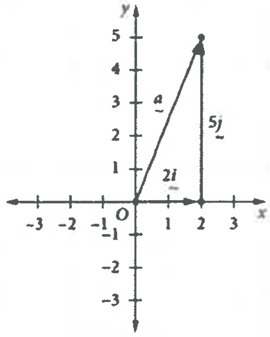
VECTORS IN COMPONENT FORM

A **unit vector** is a vector with a magnitude of one unit. To obtain a unit vector from a given vector, divide that vector by its own magnitude.

If vector $\underset{\sim}{a}$ has a magnitude of $|\underset{\sim}{a}|$, then a unit vector in the direction of $\underset{\sim}{a}$, denoted by $\hat{\underset{\sim}{a}}$, can be found by dividing vector $\underset{\sim}{a}$ by its own magnitude $|\underset{\sim}{a}|$. That is, $\hat{\underset{\sim}{a}} = \frac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$.

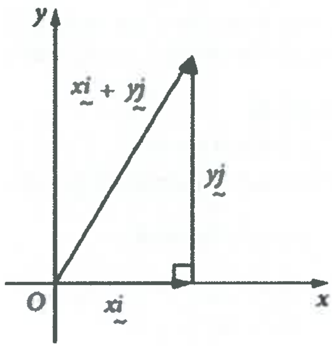
The unit vector in the direction of $\underset{\sim}{a}$ is denoted $\hat{\underset{\sim}{a}}$, where $\hat{\underset{\sim}{a}} = \frac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$ and $|\hat{\underset{\sim}{a}}| = 1$.



Recall that a vector can be represented as an ordered pair (x, y) or as a column vector $\begin{pmatrix} x \\ y \end{pmatrix}$, where the first value represents the distance parallel to the x-axis and the second value the distance parallel to the y-axis.
This information can also be defined using the vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$, where $\underset{\sim}{i}$ is a vector of magnitude one unit in the positive x-direction and $\underset{\sim}{j}$ is a vector of one unit magnitude in the positive y-direction. The vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are unit vectors.

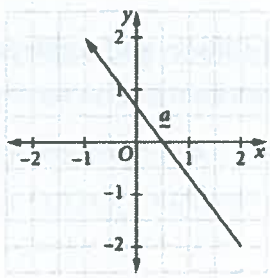
For example, the vector $\underset{\sim}{a}$ defined by the coordinates (2, 5) or the column vector $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ can be written as $\underset{\sim}{a} = 2\underset{\sim}{i} + 5\underset{\sim}{j}$.

The form $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j}$ is called **component form** or $\underset{\sim}{i}, \underset{\sim}{j}$ **form** of a vector. The vector $\underset{\sim}{a}$ may also be represented in column vector form as $\begin{pmatrix} x \\ y \end{pmatrix}$.



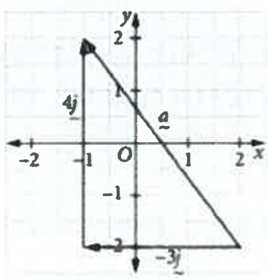
Example 10

Express the vector $\underset{\sim}{a}$ in component form.



Solution

Vectors are drawn from the tail of the vector across and then up (or down) to meet the head of the original vector, labelled as $x\underset{\sim}{i}$ and $y\underset{\sim}{j}$.
Oiginal vector in terms of the components $x\underset{\sim}{i} + y\underset{\sim}{j}$: $\quad \underset{\sim}{a} = -3\underset{\sim}{i} + 4\underset{\sim}{j}$



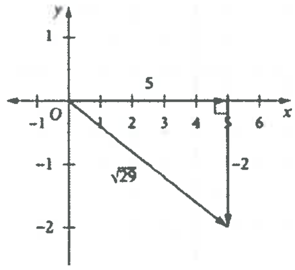
VECTORS IN COMPONENT FORM

Magnitude of a vector in component form

To find the magnitude in component form you can use Pythagoras' theorem, as the components form a right-angled triangle. For example, if $\underset{\sim}{a} = 5\underset{\sim}{i} - 2\underset{\sim}{j}$:

$$|\underset{\sim}{a}| = \sqrt{5^2 + (-2)^2}$$
$$= \sqrt{29}$$

When finding the magnitude of a vector, use only the positive square root value.

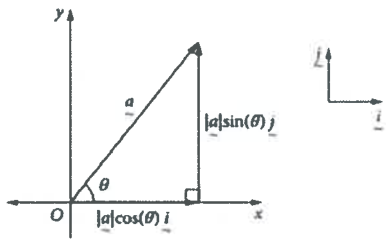


Resolving vectors into component form

If a vector $\underset{\sim}{a}$ of magnitude $|\underset{\sim}{a}|$ makes an angle θ with the positive x-axis, then: $\underset{\sim}{a} = |\underset{\sim}{a}|\cos\theta\underset{\sim}{i} + |\underset{\sim}{a}|\sin\theta\underset{\sim}{j}$.

The horizontal component of the vector $\underset{\sim}{a}$ is $|\underset{\sim}{a}|\cos\theta\,\underset{\sim}{i}$ and the vertical component is $|\underset{\sim}{a}|\sin\theta\underset{\sim}{j}$.

The process of specifying a vector of known magnitude and direction in component form is called **resolving the vector**.



Example 11

Resolve the vector $\underset{\sim}{a}$ into component form $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j}$, given $\underset{\sim}{a}$ has a magnitude of 6 units and has a direction of 50° to the positive x-axis. Give answers correct to two decimal places.

Solution

$|\underset{\sim}{a}| = 6$ and $\theta = 50°$

If a vector $\underset{\sim}{a}$ of magnitude $|\underset{\sim}{a}|$ makes an angle θ with the positive x-axis, then $\underset{\sim}{a} = |\underset{\sim}{a}|\cos\theta\underset{\sim}{i} + |\underset{\sim}{a}|\sin\theta\underset{\sim}{j}$.

$$\underset{\sim}{a} = |\underset{\sim}{a}|\cos\theta\underset{\sim}{i} + |\underset{\sim}{a}|\sin\theta\underset{\sim}{j}$$
$$= 6\cos(50°)\underset{\sim}{i} + 6\sin(50°)\underset{\sim}{j}$$
$$= 3.86\underset{\sim}{i} + 4.60\underset{\sim}{j}$$

Addition and subtraction of vectors in component form

Addition and subtraction of vectors can be done by adding or subtracting the $\underset{\sim}{i}$ components and the $\underset{\sim}{j}$ components.

For $\underset{\sim}{a} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$ and $\underset{\sim}{b} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$ then:

$$\underset{\sim}{a} + \underset{\sim}{b} = (x_1\underset{\sim}{i} + y_1\underset{\sim}{j}) + (x_2\underset{\sim}{i} + y_2\underset{\sim}{j})$$
$$= x_1\underset{\sim}{i} + x_2\underset{\sim}{i} + y_1\underset{\sim}{j} + y_2\underset{\sim}{j}$$
$$= (x_1 + x_2)\underset{\sim}{i} + (y_1 + y_2)\underset{\sim}{j}$$

Similarly:

$$\underset{\sim}{a} - \underset{\sim}{b} = (x_1\underset{\sim}{i} + y_1\underset{\sim}{j}) - (x_2\underset{\sim}{i} + y_2\underset{\sim}{j})$$
$$= (x_1 - x_2)\underset{\sim}{i} + (y_1 - y_2)\underset{\sim}{j}$$

For $\underset{\sim}{a} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$ and $\underset{\sim}{b} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$:

$$\underset{\sim}{a} + \underset{\sim}{b} = (x_1 + x_2)\underset{\sim}{i} + (y_1 + y_2)\underset{\sim}{j}$$
$$\underset{\sim}{a} - \underset{\sim}{b} = (x_1 - x_2)\underset{\sim}{i} + (y_1 - y_2)\underset{\sim}{j}$$

In column vector notation, this can be written as $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$ and $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix}$ respectively.

VECTORS IN COMPONENT FORM

Scalar multiplication of vectors in component form

$k\underset{\sim}{a} = k(x\underset{\sim}{i} + y\underset{\sim}{j})$

$= kx\underset{\sim}{i} + ky\underset{\sim}{j}$

In column vector notation, this can be written as $k\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} kx \\ ky \end{pmatrix}$.

If $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j}$, then $k\underset{\sim}{a} = kx\underset{\sim}{i} + ky\underset{\sim}{j}$

Example 12

Given $\underset{\sim}{a} = \underset{\sim}{i} - 5\underset{\sim}{j}$ and $\underset{\sim}{b} = -3\underset{\sim}{i} + 2\underset{\sim}{j}$, find: **(a)** $\underset{\sim}{a} + \underset{\sim}{b}$ **(b)** $\underset{\sim}{b} - \underset{\sim}{a}$ **(c)** $-4\underset{\sim}{a} + 7\underset{\sim}{b}$

Solution

(a) Sum of the vectors in component form: $\underset{\sim}{a} + \underset{\sim}{b} = (\underset{\sim}{i} - 5\underset{\sim}{j}) + (-3\underset{\sim}{i} + 2\underset{\sim}{j})$

Group the coefficients of the components together and simplify: $= (1-3)\underset{\sim}{i} + (-5+2)\underset{\sim}{j}$

$= -2\underset{\sim}{i} - 3\underset{\sim}{j}$

(b) Sum of the vectors in component form: $\underset{\sim}{b} - \underset{\sim}{a} = (-3\underset{\sim}{i} + 2\underset{\sim}{j}) - (\underset{\sim}{i} - 5\underset{\sim}{j})$

Group the coefficients of the components together and simplify: $= (-3-1)\underset{\sim}{i} + (2-(-5))\underset{\sim}{j}$

$= -4\underset{\sim}{i} + 7\underset{\sim}{j}$

(c) Sum of the vectors in component form: $-4\underset{\sim}{a} + 7\underset{\sim}{b} = -4(\underset{\sim}{i} - 5\underset{\sim}{j}) + 7(-3\underset{\sim}{i} + 2\underset{\sim}{j})$

$= -4\underset{\sim}{i} + 20\underset{\sim}{j} - 21\underset{\sim}{i} + 14\underset{\sim}{j}$

Group the coefficients of the components together and simplify: $= (-4-21)\underset{\sim}{i} + (20+14)\underset{\sim}{j}$

$= -25\underset{\sim}{i} + 34\underset{\sim}{j}$

Equality of vectors in component form

If $\underset{\sim}{a} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$ and $\underset{\sim}{b} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$, then $\underset{\sim}{a} = \underset{\sim}{b}$ if and only if $x_1 = x_2$ and $y_1 = y_2$.

Example 13

Find the values of m and n if $7\underset{\sim}{i} - 5\underset{\sim}{j} = (3m+1)\underset{\sim}{i} + (4n-9)\underset{\sim}{j}$.

Solution

Equate coefficients of the vector components and solve the resulting equations:

$\underset{\sim}{i}$ components: $7 = 3m + 1$

$3m = 6$

$m = 2$

$\underset{\sim}{j}$ components: $-5 = 4n - 9$

$4n = 4$

$n = 1$

VECTORS IN COMPONENT FORM

Relative position vectors

You have already looked at position vectors that represent the position of one point in relation to the origin. A **relative position vector** represents a point's position in relation to another point.

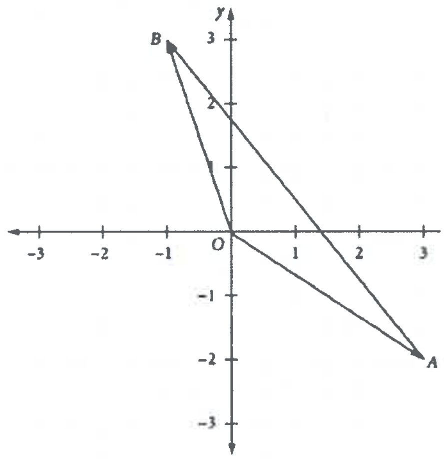
The position vector of B relative to A is given by $\overrightarrow{AB}$.

In the diagram shown $\overrightarrow{OA} = 3\underset{\sim}{i} - 2\underset{\sim}{j}$ and $\overrightarrow{OB} = -\underset{\sim}{i} + 3\underset{\sim}{j}$.

The position vector of B relative to A is $\overrightarrow{AB}$, where $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$.

Now, $\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB}$

$$\begin{aligned}\therefore\ \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA}\\ &= \left(-\underset{\sim}{i} + 3\underset{\sim}{j}\right) - \left(3\underset{\sim}{i} - 2\underset{\sim}{j}\right)\\ &= -4\underset{\sim}{i} + 5\underset{\sim}{j}\end{aligned}$$



Example 14

The position vector of point A on the Cartesian plane is $\overrightarrow{OA} = 12\underset{\sim}{i} - 5\underset{\sim}{j}$ and the position vector of point B is $\overline{OB} = -7\underset{\sim}{i} + 6\underset{\sim}{j}$. Find the position vector of A relative to B.

Solution

The position vector of A relative to B is $\overrightarrow{BA}$. Write the rule to find $\overline{BA}$: $\overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OA}$

$\therefore\ \overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$

Substitute the components and simplify: $\overrightarrow{BA} = \left(12\underset{\sim}{i} - 5\underset{\sim}{j}\right) - \left(-7\underset{\sim}{i} + 6\underset{\sim}{j}\right)$

$= 19\underset{\sim}{i} - 11\underset{\sim}{j}$

Parallel vectors

Two vectors are parallel if they are scalar multiples of each other:

If $\underset{\sim}{b} = k\underset{\sim}{a}$, where k is a real number, then $\underset{\sim}{b}$ is parallel to $\underset{\sim}{a}$.

For example, if $\underset{\sim}{a} = \underset{\sim}{i} - 3\underset{\sim}{j}$ and $\underset{\sim}{b} = 4\underset{\sim}{i} - 12\underset{\sim}{j}$, then $\underset{\sim}{b} = 4\left(\underset{\sim}{i} - 3\underset{\sim}{j}\right)$.

$\therefore\ \underset{\sim}{b} = 4\underset{\sim}{a}$, so $\underset{\sim}{b}$ is parallel to $\underset{\sim}{a}$.

Example 15

Consider the three vectors $\underset{\sim}{a} = -2\underset{\sim}{i} + 3\underset{\sim}{j}$, $\underset{\sim}{b} = 8\underset{\sim}{i} - 5\underset{\sim}{j}$ and $\underset{\sim}{c} = -8\underset{\sim}{i} + 12\underset{\sim}{j}$. Which two vectors are parallel?

Solution

Look at the vectors to see if a scalar multiplier exists for any of them: $\underset{\sim}{c} = -8\underset{\sim}{i} + 12\underset{\sim}{j}$

$$\begin{aligned}&= 4\left(-2\underset{\sim}{i} + 3\underset{\sim}{j}\right)\\ &= 4\underset{\sim}{a}\end{aligned}$$

Vectors $\underset{\sim}{a}$ and $\underset{\sim}{c}$ are parallel.

VECTORS IN COMPONENT FORM

Unit vectors in component form

Recall that a **unit vector** has a magnitude of 1 and $\hat{\underset{\sim}{a}} = \frac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$.

Therefore, if $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j}$, then $\hat{\underset{\sim}{a}} = \frac{1}{\sqrt{x^2+y^2}}\left(x\underset{\sim}{i} + y\underset{\sim}{j}\right)$.

It is usually better to express the unit vector with a rational denominator, so $\hat{\underset{\sim}{a}} = \frac{\sqrt{x^2+y^2}}{x^2+y^2}\left(x\underset{\sim}{i} + y\underset{\sim}{j}\right)$.

A unit vector is a 'direction finder', in that it determines a vector's direction but not its magnitude.

If $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j}$, **then** $\hat{\underset{\sim}{a}} = \frac{1}{\sqrt{x^2+y^2}}\left(x\underset{\sim}{i} + y\underset{\sim}{j}\right)$ **or** $\hat{\underset{\sim}{a}} = \frac{\sqrt{x^2+y^2}}{x^2+y^2}\left(x\underset{\sim}{i} + y\underset{\sim}{j}\right)$.

Example 16

Find the unit vector $\hat{\underset{\sim}{a}}$ for each of the following vectors.

(a) $\underset{\sim}{a} = 4\underset{\sim}{i} - 3\underset{\sim}{j}$ **(b)** $\underset{\sim}{a} = -5\underset{\sim}{i} + 8\underset{\sim}{j}$

Solution

(a) Divide the original vector by its magnitude to get a unit vector: $\hat{\underset{\sim}{a}} = \frac{1}{5}\left(4\underset{\sim}{i} - 3\underset{\sim}{j}\right)$

(b) Find the magnitude of the vector: $|\underset{\sim}{a}| = \sqrt{(-5)^2 + 8^2}$

$= \sqrt{89}$

Divide the original vector by its magnitude to get a unit vector: $\hat{\underset{\sim}{a}} = \frac{1}{\sqrt{89}}\left(-5\underset{\sim}{i} + 8\underset{\sim}{j}\right)$

$= \frac{\sqrt{89}}{89}\left(-5\underset{\sim}{i} + 8\underset{\sim}{j}\right)$

Unit vectors can be used to find vectors in a specified direction.

Example 17

Given $\underset{\sim}{c} = 3\underset{\sim}{i} - 6\underset{\sim}{j}$:

(a) find $\hat{\underset{\sim}{c}}$ **(b)** find vector $\underset{\sim}{d}$ of magnitude 5 in the direction of $\underset{\sim}{c}$.

Solution

(a) Find the magnitude of the vector: $|\underset{\sim}{c}| = \sqrt{3^2 + (-6)^2}$

$= \sqrt{45}$

$= 3\sqrt{5}$

Find the unit vector by dividing the vector by its magnitude: $\hat{\underset{\sim}{c}} = \frac{1}{3\sqrt{5}}\left(3\underset{\sim}{i} - 6\underset{\sim}{j}\right)$

$= \frac{\sqrt{5}}{5}\left(\underset{\sim}{i} - 2\underset{\sim}{j}\right)$

(b) Multiply the unit vector in the direction required by the required magnitude: $\underset{\sim}{d} = \frac{5\sqrt{5}}{5}\left(\underset{\sim}{i} - 2\underset{\sim}{j}\right)$

$= \sqrt{5}\left(\underset{\sim}{i} - 2\underset{\sim}{j}\right)$