MULTIPLE ROOTS OF A POLYNOMIAL EQUATION

1 The polynomial $x^3 - x^2 - 5x - 3$ has a double root at $x = \alpha$. What is the value of α?

A $-\frac{5}{3}$ **B** -1 **C** 1 **D** $\frac{5}{3}$

$(-1)^3 - (-1)^2 - 5\times(-1) - 3 = -1 - 1 + 5 - 3 = 0$ so (-1) is a root.

We can factorise by $(x+1)$

$x^3 - x^2 - 5x - 3 = (x+1)(x^2 - 2x - 3)$

$\Delta = 4 - 4\times(-3) = 16 = 4^2$ so 2 roots

$x_1 = \dfrac{2+4}{2} = 3$ and $x_2 = \dfrac{2-4}{2} = -1$

So (-1) is the double root. Response $\boxed{B}$

2 If $P(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$, then:

(a) show that $x = 1$ is a zero of multiplicity 4 **(b)** fully factorise $P(x)$.

$(x^4 - 4x^3 + 6x^2 - 4x + 1) = (x-1)(x^3 - 3x^2 + 3x - 1)$

$= (x-1)(x-1)(x^2 - 2x + 1)$

$= (x-1)(x-1)(x-1)^2$

$= (x-1)^4$

So indeed 1 is a zero of multiplicity 4

and $P(x)$ has been fully factorised.

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4 If $P(x) = x^3 - x^2 - 8x + 12$, then:

(a) show that $P(x)$ has a zero of multiplicity 2 **(b)** fully factorise $P(x)$ **(c)** solve the equation $P(x) = 0$.

a) $P(1) = 1 - 1 - 8 + 12 \neq 0$

$P(-1) = (-1)^3 - (-1)^2 - 8 \times (-1) + 12 = -1 - 1 + 8 + 12 \neq 0$

$P(2) = 2^3 - 2^2 - 8 \times 2 + 12 = 8 - 4 - 16 + 12 = 0$.

So 2 is a root of the equation, and we can factorise by $(x-2)$.

$x^3 - x^2 - 8x + 12 = (x-2)(x^2 + x - 6)$

$\Delta = 1 - 4 \times (-6) = 25 = 5^2$ so two other roots.

$x_1 = \dfrac{-1+5}{2} = 2$ and $x_2 = \dfrac{-1-5}{2} = -3$

so 2 is the double root, and $P(x)$ can be factorised as

$x^3 - x^2 - 8x + 12 = (x-2)^2(x+3)$.

The equation $P(x) = 0$ has 1 double solution: $x = 2$

and another solution $x = -3$.

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8 The polynomial $P(x) = ax^3 + bx + c$ has a multiple zero at -1 and has remainder 8 when divided by $(x - 1)$. If a, b and c are real, find their values.

$P(-1) = 0$ or $a \times (-1)^3 + b \times (-1) + c = 0$

$-a - b + c = 0$ or $a + b = c.$

So $P(x) = ax^3 + bx + a + b$

or $P(x) = a(x^3+1) + b(x+1)$

$P(x) = a(x+1)(x^2 - x + 1) + b(x+1)$

$P(x) = (x+1)\left[a(x^2-x+1) + b\right]$

Further, $P(x)$ has a multiple zero at (-1) so $\left[a(x^2-x+1)+b\right]=0$ when $x=-1$, i.e. $a((-1)^2-(-1)+1)+b=0$

or $a(1+1+1)+b=0$ $\quad 3a+b=0$

so $b=-3a$ and $c=-2a.$

$P(x) = (x+1)\left[a(x^2-x+1) - 3a\right]$

$P(x) = a(x+1)\left[x^2-x-2\right]$

$P(x) = a(x+1)(x+1)(x-2)$

The 3rd zero is $x=2.$

Now we also know $P(x)$ has a remainder of 8 when divided by $(x+1)$

So $P(+1) = 8$ $\quad P(-1) = a(2) \times 2 \times (1-2) = -4a$

so $-4a = 8$ $\quad \boxed{a = -2}$

$P(x) = -2(x+1)^2(x-2)$

MULTIPLE ROOTS OF A POLYNOMIAL EQUATION

9 Solve each equation using the properties of polynomials.

(a) $4x^3 - 8x^2 + 5x - 1 = 0$, given that it has a root of multiplicity 2.

(b) $x^4 + 4x^3 - 16x - 16 = 0$, given that it has a root of multiplicity 3.

a) $P(1) = 4 - 8 + 5 - 1 = -4 + 4 = 0$ so 1 is a root.

$4x^3 - 8x^2 + 5x - 1 = (x-1)(4x^2 - 4x + 1)$

$\Delta = 16 - 4 \times 4 = 0$ so 1 double root $x = \frac{4}{2 \times 4} = \frac{1}{2}$

$P(x) = 4(x-1)\left(x - \frac{1}{2}\right)^2$

b) $P(1) = 1^4 + 4 \times 1^3 - 16 \times 1 - 16 \neq 0$

$P(-1) = (-1)^4 + 4 \times (-1)^3 - 16 \times (-1) - 16 = 1 - 4 + 16 - 16 \neq 0$

$P(-2) = (-2)^4 + 4(-2)^3 - 16 \times (-2) - 16 = 16 - 32 + 32 - 16 = 0$

so (-2) is a root, and we can factorise by $(x+2)$.

$x^4 + 4x^3 - 16x - 16 = (x+2)(x^3 + 2x^2 - 4x - 8)$

Now we look for roots of $x^3 + 2x^2 - 4x - 8$

Let's see if (-2) is also a root of this polynomial.

$Q(-2) = (-2)^3 + 2(-2)^2 - 4 \times (-2) - 8 = -8 + 8 + 8 - 8 = 0$

So we can factorise further.

$P(x) = (x+2)(x+2)(x^2 - 4)$

$P(x) = (x+2)^3(x-2)$

(-2) is the root of multiplicity 3

2 is the other root, of multiplicity 1