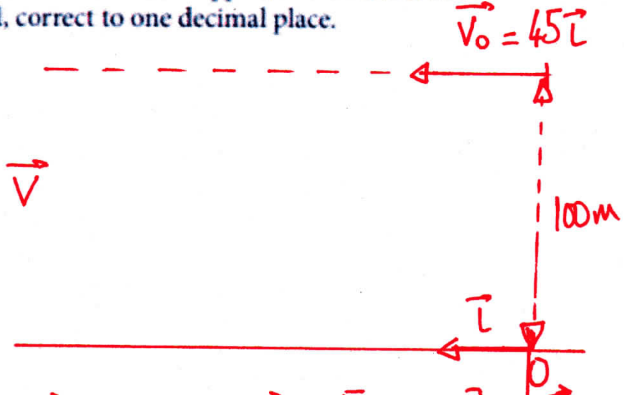
PROJECTILES AND RESISTED MOTION

1 An aircraft drops a package of emergency rations to a family stranded in the floods. The aircraft is travelling horizontally at $45.0\,\mathrm{m\,s^{-1}}$ and is 100 m above the ground. A parachute allows the package to fall with constant speed and hit the ground 10 s after release. Air resistance can be ignored.

(a) Find where the package hits the ground relative to the point from where it was dropped, to the nearest metre.

(b) Find the velocity of the package just before it hits the ground, correct to one decimal place.



a) $\vec{V} = V_0 \vec{i} + V_v \vec{j}$

where V_v is the vertical component of $\vec{V}$

So $\vec{r}(t) = V_0 t \vec{i} + V_v t \vec{j} + \vec{C}$

At $t=0$ $\vec{r}(0) = -100\vec{j}$ so $\vec{r}(t) = V_0 t \vec{i} + [V_v t - 100]\vec{j}$

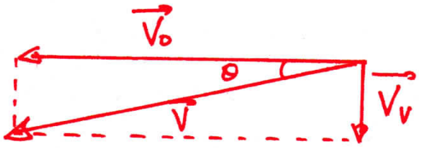
So $\begin{cases} x(t) = V_0 t \\ y(t) = V_v t - 100 \end{cases}$

The package hits the ground after 10 s, so $y(10) = 0 = V_v \times 10 - 100$

so $V_v = \dfrac{100}{10} = 10\ \mathrm{m\,s^{-1}}$

When $t = 10$ s, $x(10) = 45 \times 10 = 450$ m.

b) At $t = 10$ s, $\vec{V} = 45\vec{i} + 10\vec{j}$



so $|\vec{V}| = \sqrt{45^2 + 10^2} = 46.1\ \mathrm{m\,s^{-1}}$

with $\tan\theta = \dfrac{V_v}{V_0} = \dfrac{10}{45} = \dfrac{2}{9}$ so $\theta \simeq 12.5°$ approx.

2 A particle is projected from level ground with a velocity of $7\underset{\sim}{i} + 24\underset{\sim}{j}\ \mathrm{m\,s^{-1}}$, where $\underset{\sim}{i}$ is horizontal and $\underset{\sim}{j}$ is vertically up.

(a) Find the initial speed and angle of projection of the particle. Give the angle of projection, correct to the nearest tenth of a degree. Air resistance can be ignored.

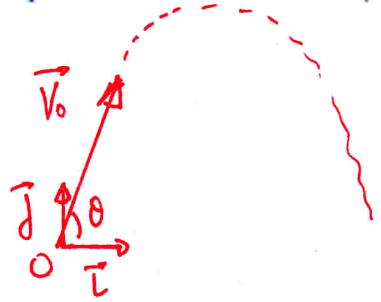
(b) Find the time of flight of the particle. Give your answer correct to two decimal places.

(c) Find the horizontal distance travelled by the particle. Give your answer correct to one decimal place.

(d) Find the maximum height reached by the particle. Give your answer correct to one decimal place.

(e) Determine whether the particle is ever travelling in a direction perpendicular to its initial velocity.

a) $\vec{V_0} = 7\vec{i} + 24\vec{j}$



$|\vec{V_0}| = \sqrt{7^2 + 24^2} = 25\ m s^{-1}$

$\tan\theta = \frac{24}{7}$ so $\theta \simeq 73.7\,°$

b) $\vec{a} = -g\vec{j}$ so $\vec{V}(t) = -gt\vec{j} + \vec{C}$

At $t = 0$ $\vec{V}(0) = \vec{V_0}$ so $\vec{C} = \vec{V_0} = 7\vec{i} + 24\vec{j}$

So $\vec{V}(t) = [-gt + 24]\vec{j} + 7\vec{i}$.

So $\vec{r}(t) = 7t\,\vec{i} + [24t - \frac{1}{2}gt^2]\,\vec{j} + \vec{K}$

At $t = 0$, $\vec{r}(0) = \vec{0}$, so $\vec{K} = \vec{0}$ $\therefore$ $\vec{r}(t) = \underbrace{7t}_{=x(t)}\vec{i} + \underbrace{[24t - \frac{1}{2}gt^2]}_{=y(t)}\vec{j}$

So $y(t) = 0$ when either $t = 0$ or when $24 - \frac{1}{2}gt = 0$

i.e. $t = \frac{48}{g} = 4.90$ s approx.

c) when $t = \frac{48}{g}$ $x\left(\frac{48}{g}\right) = 7 \times \frac{48}{g} = 34.3$ m

d) The maximum height is reached when $\vec{V}_{vertical} = \vec{0}$, i.e.

$-gt + 24 = 0 \Leftrightarrow t = \frac{24}{g} = 2.45$ s.

For $t = 24/g$ $y(24/g) = \frac{24}{g}\left(24 - \frac{1}{2}g \times \frac{24}{g}\right) = 29.4$ m approx.

e) $\vec{V}(t) \cdot \vec{V_0} = 0$ when $7 \times 7 + 24(-gt + 24) = 0$

$\Leftrightarrow 625 - 24gt = 0 \Leftrightarrow t = \frac{625}{24g} \simeq 2.66$ s.

At $t = 2.66$ s $\vec{V} \perp \vec{V_0}$

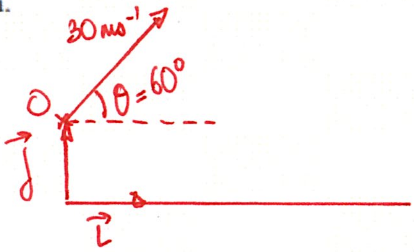
PROJECTILES AND RESISTED MOTION

3 A cricket ball is thrown from a height of 1 metre with a speed of $30\,\text{m s}^{-1}$ and at an angle of 60° to the horizontal.

(a) Taking the origin at the point of projection and assuming the ground to be level, find the horizontal distance travelled by the ball before it lands. Give your answer correct to one decimal place.

(b) Find the maximum height of the ball above the ground. Give your answer correct to one decimal place.

(c) Find the Cartesian equation of the path of the ball and sketch its path.



a) $\vec{a} = -g\vec{j}$

so $\vec{v}(t) = -gt\vec{j} + \vec{C}$

At $t=0$ $\quad \vec{v}(0) = V_0 \cos 60\,\vec{i} + V_0 \sin 60\,\vec{j}$

i.e. $\vec{v}(0) = \frac{V_0}{2}\vec{i} + \frac{V_0\sqrt{3}}{2}\vec{j}$ $\quad$ so $\vec{v}(t) = \left(\frac{V_0}{2}\right)\vec{i} + \left[\frac{V_0\sqrt{3}}{2} - gt\right]\vec{j}$

So $\vec{r}(t) = \left[\frac{V_0 t}{2}\right]\vec{i} + \left[\frac{V_0\sqrt{3}}{2}t - \frac{1}{2}gt^2\right]\vec{j} + \vec{K}$

At $t=0$ $\quad \vec{r}(0) = \vec{0}$ $\quad$ so $\vec{r}(t) = \underbrace{\left[\frac{V_0 t}{2}\right]}_{x(t)}\vec{i} + \underbrace{\left[\frac{V_0\sqrt{3}t}{2} - \frac{1}{2}gt^2\right]}_{y(t)}\vec{j}$

so $x(t) = \frac{V_0 t}{2} \iff t = \frac{2x}{V_0}$

When $y=-1$, then $\frac{V_0\sqrt{3}}{2}t - \frac{gt^2}{2} = -1 \iff gt^2 - V_0\sqrt{3}t - 2 = 0$

$\Delta = (V_0\sqrt{3})^2 - 4 \times g \times (-2) = 3V_0^2 + 8g = 3\times 30^2 + 8\times 10 = 2780$

So $t = \frac{V_0\sqrt{3} + \sqrt{2780}}{2g} = \frac{30\sqrt{3}+\sqrt{2780}}{20} = 5.23\,s.$

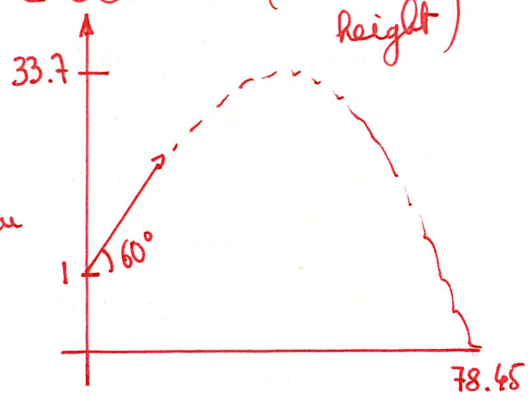
At $t = 5.23\,s$, $\quad x = \frac{30 \times 5.23}{2} = 78.45$ m which is the range.

b) the vertical speed is zero when $t = \frac{V_0\sqrt{3}}{2g} = 2.62\,s$

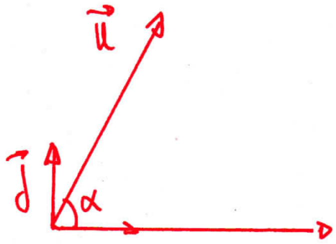
At that time, $y = \frac{30\sqrt{3}}{2} \times 2.62 - \frac{10\times(2.62)^2}{2} = 33.7$ m (maximum height)

c) $y = \frac{\cancel{V_0}\sqrt{3}}{\cancel{2}} \times \frac{\cancel{2}x}{\cancel{V_0}} - \frac{g}{2}\left(\frac{2x}{V_0}\right)^2$

So $y = \sqrt{3}x - \frac{2gx^2}{V_0^2}$ is the path's Cartesian equation



PROJECTILES AND RESISTED MOTION



7 A projectile is projected from the origin with initial speed u and angle of projection α.

(a) Show that the Cartesian equation of its path is $y = x\tan\alpha - \frac{gx^2}{2u^2}\sec^2\alpha$.

(b) Hence find an expression for the projectile's range R.

(c) Hence find an expression for the projectile's maximum height y_{max}.

(d) Show that for a given initial speed, the maximum range of the projectile occurs when $\alpha = 45°$ and is given by $R = \frac{u^2}{g}$.

a) $\vec{a} = -g\vec{j}$ so $\vec{V} = -gt\vec{j} + \vec{C}$

At $t=0$ $\vec{V}(0) = \vec{u} = u\cos\alpha\,\vec{i} + u\sin\alpha\,\vec{j}$

So $\vec{v}(t) = u\cos\alpha\,\vec{i} + [u\sin\alpha - gt]\vec{j}$

So $\vec{r}(t) = u\cos\alpha\, t\,\vec{i} + [u\sin\alpha\, t - \frac{1}{2}gt^2]\vec{j} + \vec{K}$.

At $t=0$, $\vec{r}(0) = 0$, so $\vec{K} = \vec{0}$

So $\vec{r}(t) = u\cos\alpha\, t\,\vec{i} + [u\sin\alpha\, t - \frac{1}{2}gt^2]\vec{j}$

$$\begin{cases} x(t) = u\cos\alpha\, t \iff t = \dfrac{x}{u\cos\alpha} \\ y(t) = u\sin\alpha\, t - \frac{1}{2}gt^2 \end{cases}$$

so $y = \cancel{u}\sin\alpha \dfrac{x}{\cancel{u}\cos\alpha} - \frac{1}{2}g\left[\dfrac{x}{u\cos\alpha}\right]^2$

$$y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2u^2}$$

b) Range is when $y=0$, i.e. $\dfrac{gR\sec^2\alpha}{2u^2} = \tan\alpha \Rightarrow R = \dfrac{2u^2\tan\alpha}{g\sec^2\alpha}$

$R = \dfrac{2u^2}{g}\dfrac{\sin\alpha}{\cos\alpha} \times \cos^2\alpha = \dfrac{u^2}{g} \times 2\sin\alpha\cos\alpha = \dfrac{u^2\sin 2\alpha}{g}$

c) $\vec{V_v} = \vec{0}$ when $t = \dfrac{u\sin\alpha}{g}$

For $t = \dfrac{u\sin\alpha}{g}$, $y = u\sin\alpha \times \dfrac{u\sin\alpha}{g} - \frac{1}{2}g\left(\dfrac{u\sin\alpha}{g}\right)^2$

$y = \dfrac{u^2\sin^2\alpha}{g} - \dfrac{u^2\sin^2\alpha}{2g} = \dfrac{u^2\sin^2\alpha}{2g}$

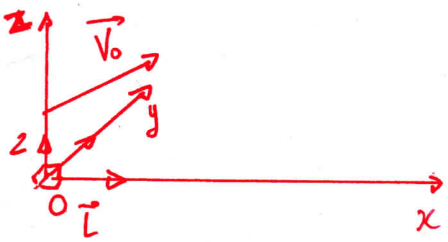
d) R is a function of α

$R'(\alpha) = \dfrac{2u^2}{g}\cos 2\alpha$.

the maximum occurs when $R'(\alpha) = 0$, i.e. when $\cos 2\alpha = 0$ so $\alpha = 45°$

For $\alpha = 45$, $R = u^2/g$.

9 The velocity (in m s^{-1}) at time t seconds of a ball hit from a height of 2 metres above ground level is given by $\dot{\underset{\sim}{r}} = 12\underset{\sim}{i} + 9\underset{\sim}{j} + (30 - 9.8t)\underset{\sim}{k}$, where $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit vectors in the east, north and vertically up directions respectively and the origin is at ground level. Find the ball's height above ground level, in metres, after 2 seconds.



$$\vec{\dot{r}}(t) = 12\vec{i} + 9\vec{j} + (30 - 9.8t)\vec{k}$$

$$\vec{r}(t) = 12t\vec{i} + 9t\vec{j} + \left(30t - \frac{9.8}{2}t^2\right)\vec{k} + \vec{C}$$

At $t = 0$ $\quad \vec{r}(0) = 2\vec{k}$ $\quad$ so $\vec{C} = 2\vec{k}$

$$\vec{r}(t) = \underbrace{12t}_{x(t)}\vec{i} + \underbrace{9t}_{y(t)}\vec{j} + \underbrace{\left[30t - \frac{9.8}{2}t^2 + 2\right]}_{z(t)}\vec{k}$$

So At $t = 2$ s, $\quad z(2) = 30 \times 2 - \frac{9.8}{2} \times 2^2 + 2$

$z(2) = 42.4$ m

At $t = 2$ s, the ball is 42.4 m above the ground.

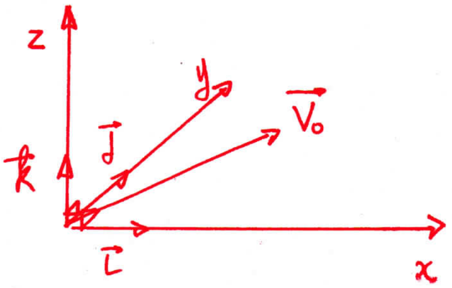
10 A golf ball is hit from the ground with a velocity $20\underset{\sim}{i}+0\underset{\sim}{j}+15\underset{\sim}{k}$ m s^{-1}, where $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit vectors horizontally forward, horizontally to the left, and vertically upwards, respectively. After being hit, the ball has a gravitational acceleration of $-10\underset{\sim}{k}$ m s^{-2} and also has a 'hook' (i.e. a horizontal acceleration to the left) of $4\underset{\sim}{j}$ m s^{-2}. Air resistance can be ignored.

(a) Find the expression for $\underset{\sim}{r}$, the position vector of the ball at time t.

(b) When the ball hits the ground, how far will it be to the left of the line along the horizontally forward direction?

(c) Find when the speed of the ball is a minimum. Give your answer correct to two decimal places.

(d) Calculate the minimum speed. Give your answer correct to one decimal place.



$\vec{V_0} = 20\vec{i} + 15\vec{k}$

$\vec{a} = -10\vec{k} + 4\vec{j}$

a) $\vec{V}(t) = -10t\,\vec{k} + 4t\,\vec{j} + \vec{C}$

At $t=0$ $\vec{V}(0) = \vec{V_0} = 20\vec{i} + 15\vec{k}$ so $\vec{C} = 20\vec{i} + 15\vec{k}$

$\therefore\ \vec{V}(t) = 20\vec{i} + 4t\,\vec{j} + [15 - 10t]\,\vec{k}$

$\therefore\ \vec{r}(t) = 20t\,\vec{i} + 2t^2\,\vec{j} + [15t - 5t^2]\,\vec{k} + \vec{K}$

At $t=0$ $\vec{K} = \vec{0}$ so $\vec{r}(t) = 20t\,\vec{i} + 2t^2\,\vec{j} + 5(3t - t^2)\,\vec{k}$

b) When $z=0$, $t = 3$ s; at that time, $y(3) = 2\times 3^2 = 18$ m.
So the ball will be 18 m to the left of the line.

c) $|\vec{V}(t)| = \sqrt{20^2 + (4t)^2 + (15-10t)^2} = \sqrt{400 + 16t^2 + 225 + 100t^2 - 300t}$

$|\vec{V}(t)| = \sqrt{116t^2 - 300t + 625}$ so $V'(t) = \dfrac{1}{2} \times \dfrac{(232t - 300)}{\sqrt{116t^2 - 300t + 625}}$

$V(t)$ is minimum for $232t - 300 = 0$, i.e. $t = 300/232 \approx 1.29$ s

d) When $t = \dfrac{300}{232} = \dfrac{75}{58}$ $\quad V(t) = \sqrt{116\times\left(\dfrac{75}{58}\right)^2 - 300\times\dfrac{75}{58} + 625}$

$V\left(\dfrac{75}{58}\right) = 20.8$ m s^{-1}

12 The velocity (in m s^{-1}) at time t seconds of a ball thrown at a height of 12 metres above ground level is given by $\dot{\underset{\sim}{r}} = 8\underset{\sim}{i} + 3\underset{\sim}{j} + (20 - 9.8t)\underset{\sim}{k}$, where $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit vectors in the east, north and vertically up directions respectively and the origin is at ground level. Find when the ball hits the ground.

$$\vec{v}(t) = 8\vec{i} + 3\vec{j} + (20 - 9.8t)\vec{k}$$

$$\vec{r}(t) = 8t\,\vec{i} + 3t\,\vec{j} + \left(20t - \frac{9.8}{2}t^2\right)\vec{k} + \vec{C}$$

At $t = 0$ $\quad \vec{r}(0) = 12\vec{k}$

So $\vec{r}(t) = 8t\,\vec{i} + 3t\,\vec{j} + \underbrace{\left[4.9t^2 + 20t + 12\right]}_{z(t)}\vec{k}$

the ball hits the ground when $z(t) = 0$, i.e.

$$-4.9t^2 + 20t + 12 = 0$$

$$\Delta = 400 + 4 \times 12 \times 4.9$$

$$\Delta = 635.2$$

$$t = \frac{-20 - \sqrt{635.2}}{2 \times (-4.9)}$$

$$t = 4.61 \text{ s.}$$

The ball hit the ground at $t = 4.61$ s.

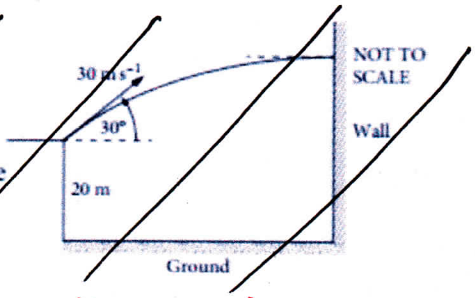
PROJECTILES AND RESISTED MOTION

13 The trajectory of a projectile fired with speed u m s^{-1} at an angle θ to the horizontal, in a medium whose resistance to the projectile's motion is proportional to the projectile's velocity, is represented by the parametric equations $x = \dfrac{u\cos\theta}{k}\left(1-e^{-kt}\right)$ and $y = \dfrac{(10+ku\sin\theta)}{k^2}\left(1-e^{-kt}\right)-\dfrac{10t}{k}$, where k is the constant of proportionality of the resistance.

(a) Show that the greatest height is reached when $t = \dfrac{1}{k}\log_e\left(\dfrac{10+ku\sin\theta}{10}\right)$.

(b) Find the greatest height.

~~A ball is thrown from a point 20 m above the horizontal ground in the same medium as mentioned above. It is thrown with speed 30 m s^{-1} at an angle of 30° to the horizontal. At its highest point it hits a wall as shown in the diagram.~~



~~**(c)** If $k = 0.4$, show that the ball hits the wall at a height of 25.8 m above the ground.~~

~~**(d)** What is the horizontal distance of the wall from the point of projection?~~

a) The greatest height is reached when $\dot{y}(t) = 0$, i.e.

$$\dot{y}(t) = \left(\frac{10+ku\sin\theta}{k^{\cancel{2}}}\right)\times\left(+\cancel{k}e^{-kt}\right)-\frac{10}{k}$$

So $\dot{y}(t) = 0 \iff \dfrac{(10+ku\sin\theta)}{\cancel{k}}e^{-kt} = \dfrac{10}{\cancel{k}}$

$$\iff e^{-kt} = \frac{10}{10+ku\sin\theta} \iff t = \frac{1}{k}\ln\left(\frac{10+ku\sin\theta}{10}\right)$$

b) When $t = \dfrac{1}{k}\ln\left(\dfrac{10+ku\sin\theta}{10}\right)$ $\quad\left(\text{OR } e^{-kt} = \dfrac{10}{10+ku\sin\theta}\right)$

then $y = \left(\dfrac{10+ku\sin\theta}{k^2}\right)\times\left[1-\dfrac{10}{10+ku\sin\theta}\right]-\dfrac{10}{k}\times\dfrac{1}{k}\ln\left(\dfrac{10+ku\sin\theta}{10}\right)$

$$y = \left(\frac{10+ku\sin\theta}{k^2}\right)\left[\frac{10+ku\sin\theta-10}{10+ku\sin\theta}\right]-\frac{10}{k^2}\ln\left(\frac{10+ku\sin\theta}{10}\right)$$

$$y = \frac{u\sin\theta}{k}-\frac{10}{k^2}\ln\left[\frac{10+ku\sin\theta}{10}\right]$$

14 A projectile is fired from the origin O with an initial velocity $V\ \mathrm{m\,s^{-1}}$ at an angle θ to the horizontal in a medium whose resistance is proportional to the velocity.

The parametric equations of the trajectory are $x = \frac{V\cos\theta}{k}\left(1-e^{-kt}\right)$ and $y = \frac{(10+kV\sin\theta)}{k^2}\left(1-e^{-kt}\right)-\frac{10t}{k}$, where $k = 0.2$ is the constant of proportionality of the resistance.

The projectile is fired at an angle of 45° to the horizontal with an initial velocity of $20\sqrt{2}\ \mathrm{m\,s^{-1}}$.

(a) Find when the projectile reaches its greatest height, correct to one decimal place.
(b) Find the greatest height attained, correct to one decimal place.
(c) Show that the projectile hits the ground when $t \approx 3.6$ s **(i)** graphically **(ii)** by substitution.
(d) Find the horizontal range of the projectile.

a) $x(t) = \frac{V\cos\theta}{k}\left(1-e^{-kt}\right)$ $\qquad y(t) = \frac{(10+kV\sin\theta)}{k^2}\left(1-e^{-kt}\right) - \frac{10t}{k}$

the greatest height is reached when $\dot{y}(t) = 0$, i.e.

$$\frac{(10+kV\sin\theta)}{\cancel{k^2}}\left(+\cancel{k}e^{-kt}\right) - \frac{10}{\cancel{k}} = 0$$

$$\Leftrightarrow (10+kV\sin\theta)\,e^{-kt} = 10$$

$$\Leftrightarrow e^{-kt} = \frac{10}{10+kV\sin\theta} \quad \Leftrightarrow t = \frac{1}{k}\ln\left[\frac{10+kV\sin\theta}{10}\right]$$

$$\Leftrightarrow t = \frac{1}{k}\ln\left[1+\frac{kV\sin\theta}{10}\right]$$

So $t = \frac{1}{0.2}\ln\left[1+\frac{0.2\times 20\sqrt{2}\sin 45}{10}\right] = 5\ln\left[1+0.4\sqrt{2}\frac{\sqrt{2}}{2}\right] = 5\ln 1.4$
$\simeq 1.7$ s.

b) For $t = 5\ln 1.4$,

$$y(5\ln 1.4) = \frac{(10+0.2\times 20\sqrt{2}\sin 45)}{0.2^2}\left(1-e^{-0.2\times 5\ln 1.4}\right) - \frac{10\times 5\ln 1.4}{0.2}$$

$$y(5\ln 1.4) = \left(\frac{10+4}{0.2^2}\right)\left(1-\frac{1}{1.4}\right) - 250\ln 1.4$$

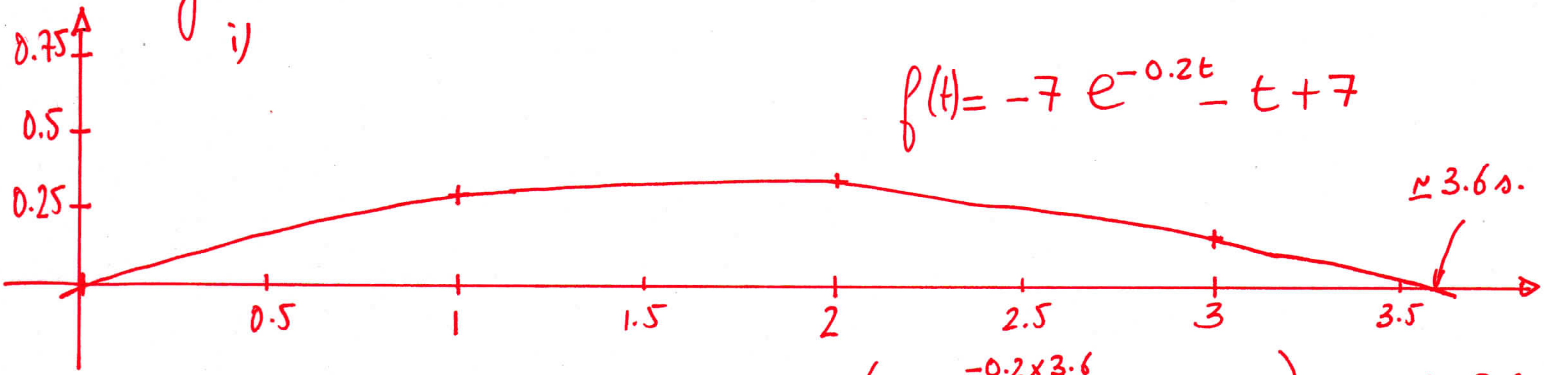
$$y(5\ln 1.4) = 350\left(\frac{2}{7}\right) - 250\ln 1.4 = 100 - 250\ln 1.4 \simeq 15.9\text{ m}$$

c) $y(t) = \dfrac{(10 + 0.2 \times 20\sqrt{2} \sin 45)}{0.2^2}\left(1 - e^{-0.2t}\right) - \dfrac{10t}{0.2}$

$y(t) = 350\left(1 - e^{-0.2t}\right) - 50t$

$y(t) = -350 e^{-0.2t} - 50t + 350 = 50\left(-7e^{-0.2t} - t + 7\right)$

i)



ii) when $t = 3.6$ s $\quad y(3.6) = 50\left(-7e^{-0.2 \times 3.6} - 3.6 + 7\right) = -0.36$

so close to zero.

d) At $t = 3.6$ s, $\quad x(3.6) = \dfrac{V\cos\theta}{k}\left(1 - e^{-k \times 3.6}\right)$

At that point, $\quad 1 - e^{-0.2 \times 3.6} = \dfrac{50 \times 3.6}{350}$

so $\quad x(3.6) = \dfrac{20\sqrt{2}\cos 45}{0.2} \times \left(\dfrac{50 \times 3.6}{350}\right)$

$x(3.6) = 100 \times \dfrac{18}{35} = \dfrac{1800}{35} = \dfrac{360}{7} \simeq 51.4$ m