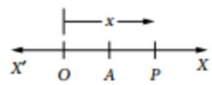
VELOCITY AND ACCELERATION AS A RATE OF CHANGE

Particle is the term used for a body that behaves such that all forces acting on the body can be regarded as acting through a single point. This means that you can represent the body as a single point, regardless of its actual size and shape. This definition of a particle means that quite large bodies, e.g. trains, can still be classified as 'particles' provided this condition applies.

Displacement

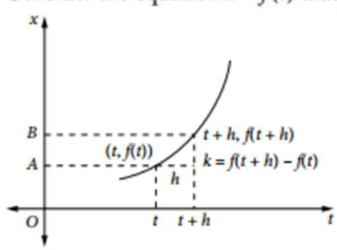


Consider a particle, which can be represented by a point P, moving in a straight line $X'OX$. The **displacement** x is the particle's position relative to the fixed point O. It may be a positive or negative number, according to whether P is to the right or left of O. The origin of the motion is not necessarily at O, so when $t = 0$, P may be (for example) at the point A.

Displacement is defined as the position relative to a starting point. It can be positive or negative.
Displacement does not necessarily represent the total distance travelled.
Unlike displacement, distance is always a positive quantity.

Velocity

Consider the equation $x = f(t)$ that gives the position coordinate x of a particle moving in a straight line at time t.



At time t, the particle is at A, and at time $(t + h)$ the particle is at B, as shown in the diagram. Thus in the small time interval h the particle has changed its position by an amount $k = f(t + h) - f(t)$.

The average **velocity** in this time interval $= \dfrac{k}{h} = \dfrac{f(t+h) - f(t)}{h}$, $h \neq 0$.

The instantaneous velocity of the particle at time t is defined by

$\lim\limits_{h \to 0} \dfrac{f(t+h) - f(t)}{h}$. It may be denoted by $v(t)$, $f'(t)$, $\dfrac{dx}{dt}$ or $\dot{x}$.

Velocity is defined as the rate of change of position (i.e. of displacement) with respect to time, or as the time rate of change of position in a given direction.

$$v(t) = f'(t) = \frac{dx}{dt} = \dot{x} = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

Velocity can be positive or negative, depending on the direction of travel.
Speed is the magnitude of the velocity and is always positive.

Example 31

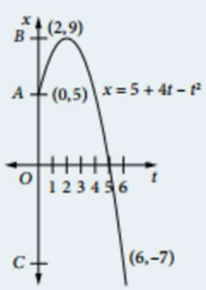
Consider the equation $x = 5 + 4t - t^2$, which defines the displacement x metres from O at time t seconds (for $t \geq 0$) of a particle moving in a straight line.

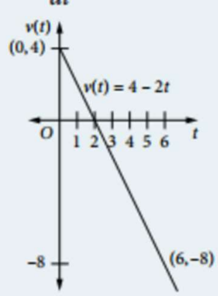
(a) Find the velocity function. **(b)** Discuss the sign of the velocity over $0 \leq t \leq 6$.

Solution

(a) For $x = 5 + 4t - t^2$: $\quad v = \dfrac{dx}{dt} = 4 - 2t$

(b)





The diagram at left is the graph of the displacement function. The diagram at right is the graph of the velocity function.

- When $t = 0$, $v = 4$ and the particle is at point A.
- When $t = 2$, $v = 0$ and the particle is at rest at B. For the first 2 seconds of motion the particle is moving in a positive direction with a positive velocity.
- When $t = 3$, $v = -2$. This negative velocity means that the particle is moving in a negative direction (i.e. 'backwards') with a speed of $2\,\text{m s}^{-1}$.
- When $t = 6$, $v = -8$. The particle is moving with a speed of $8\,\text{m s}^{-1}$ in the negative direction and is at the point C.

Over $0 \le t \le 6$, the particle moves from A to B to C. Its final displacement from O is -7 m. The total distance travelled is 20 m, which includes the distance from A to B (4 m) plus the distance from B to C (16 m).

After the particle was at rest (at $t = 2$ seconds), it reversed its direction and was then moving in the opposite direction.

Also, when $t = 3$, $v = -2$ and $x = 8$. The particle has moved from a displacement of 9 m to a displacement of 8 m. The displacement has decreased, so its rate of change (i.e. velocity) is negative.

Acceleration

Acceleration is defined as the rate of change of velocity with respect to time. Acceleration, like velocity, can be positive or negative. Positive acceleration indicates that the velocity is increasing, while negative acceleration indicates that the velocity is decreasing, which is often called deceleration or retardation.

If you denote the velocity by $v(t)$, then the average acceleration over the time interval from t to $(t + h)$ is $\dfrac{v(t+h)-v(t)}{h}$

The instantaneous acceleration at time t is defined by $\lim\limits_{h\to 0}\dfrac{v(t+h)-v(t)}{h}$. It may be denoted by $v'(t)$, $a(t)$, $f''(t)$, $\dfrac{dv}{dt}$, $\dfrac{d^2x}{dt^2}$ or $\ddot{x}$:

$$a(t) = v'(t) = \frac{d^2x}{dt^2} = \ddot{x} = \lim_{h\to 0}\frac{v(t+h)-v(t)}{h}$$

Example 32

A particle is moving in a straight line such that its displacement x m from a fixed point O on the line at time t seconds (for $t \ge 0$) is given by $x = t^3 - 12t + 16$. Find:

(a) the particle's initial displacement, velocity and acceleration

(b) the time when its velocity is zero, and its displacement and acceleration at this time.

Solution

(a) $x = t^3 - 12t + 16$

$v = \dfrac{dx}{dt} = 3t^2 - 12$

$a = \dfrac{dv}{dt} = 6t$

When $t = 0$: $x = 16$, $v = -12$, $a = 0$

Initially the particle is 16 m from O, moving with a velocity of -12 m s^{-1} and with zero acceleration.

(b) $v = 0$: $3t^2 - 12 = 0$

Factorise: $3(t-2)(t+2) = 0$

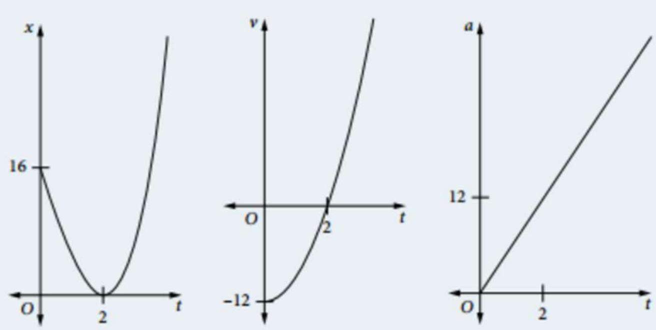
$\therefore t = 2, -2$

But $t \ge 0$, so the particle is at rest at 2 seconds.

When $t = 2$: $x = 8 - 24 + 16 = 0$, so the particle is at O.

$a = 12$, so acceleration is 12 m s^{-2}.

It is worth looking at the graphs of the functions for x, v and a:



From the graphs we can see that the acceleration, after initially being zero, is positive and increasing at a constant rate. As $a > 0$, the velocity is increasing at all times. During the first two seconds the particle moves from 16 metres on the positive side of O back to O, with an increasing negative velocity. After 2 seconds the particle is at rest at O ($v = 0$). It then moves back in a positive direction with an increasing velocity.

'initially':	$t = 0$	'at the origin':	$x = 0$
'at rest':	$v = 0$	'velocity is constant':	$a = 0$