# POLYNOMIAL FUNCTIONS

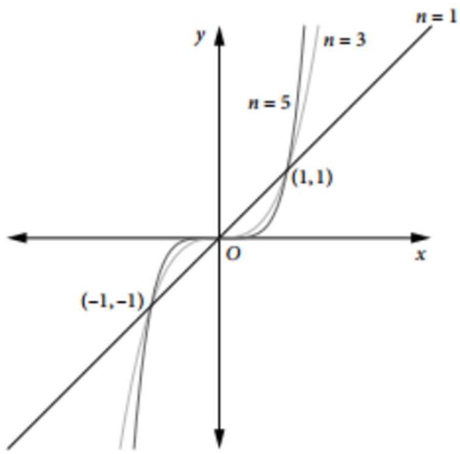
The function $f$, where $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ $(a_n \neq 0)$, is called the **general polynomial function**. This function is defined for all real values of $x$ and is continuous and differentiable. Following the earlier notation, $P(x)$ is often written instead of $f(x)$ when defining rules for polynomial functions. For $a \neq 0$, there is:

- $f(x) = ax + b$ (general linear function)
- $f(x) = ax^2 + bx + c$ (general quadratic function)
- $f(x) = ax^3 + bx^2 + cx + d$ (general cubic function)

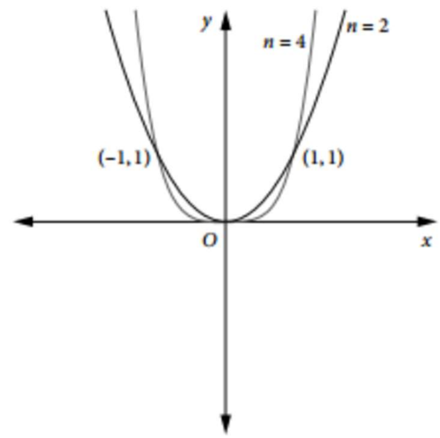
If the coefficient of the highest power of $x$ is unity (1), the polynomial is said to be monic.

## Graphs of polynomial functions

The simplest polynomial function of degree $n$ is $f(x) = x^n$. Graphs of these basic polynomials for $n = 1, 2, 3, 4, 5$ are shown below (with $n = 1, 3, 5$ on the left and $n = 2, 4$ on the right).




Odd functions $f(-x) = -f(x)$




Even functions $f(-x) = f(x)$

## Important features of $f(x) = x^n$

1. The $x$-axis is a tangent to each graph at the origin ($n \neq 1$).
2. $f(x) = x^n$ for even values of $n$ defines **even functions** (see above right).
   For even functions, $f(-x) = f(x)$, so their graphs are symmetrical about the $y$-axis.
   For example, if $f(x) = x^4$, $f(-x) = (-x)^4 = x^4 = f(x)$.
3. $f(x) = x^n$ for odd values of $n$ defines **odd functions** (see above left).
   For odd functions, $f(-x) = -f(x)$. Because $f(-x)$ and $f(x)$ are opposite in sign, the graph of $f$ for $x \leq 0$ can be obtained by rotating the graph of $f$ for $x \geq 0$ through 180° about the origin.
   For example, if $f(x) = x^3$, $f(-x) = (-x)^3 = -x^3 = -f(x)$.
4. Recognising that a function is odd or even means that you only need to draw half of the graph in detail. The other half can then be drawn using the symmetry properties.
5. Odd functions will have an inverse function, although some may require a restriction on the domain.
6. Even functions will not have a single inverse function, but can be split into two parts (by restricting the domain), so that each part has an inverse function. (Inverse functions are covered in Chapter 5.)
7. Note that most functions are neither even nor odd, e.g. $f(x) = x^2 + x$, $f(x) = e^x$.

## Graphs and graphing software

It is a good idea to use graphing software to check your sketches. Software-generated graphs will also allow you to zoom in and to easily find the coordinates of important points.

### Example 16

Sketch the graph of the following.

**(a)** $f(x) = x^3$ **(b)** $f(x) = (x-2)^3$ **(c)** $f(x) = (x-2)^3 + 1$

**Solution**

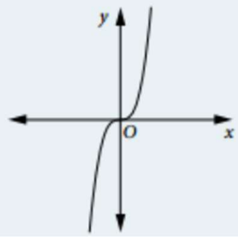
**(a)** $f(x) = x^3 = 0$ for $x = 0$

For all $x < 0, f(x) < 0$

For all $x > 0, f(x) > 0$

$f(-a) = (-a)^3, f(a) = a^3$

$\therefore\ f(-x) = -f(x)$, so the function is odd.




**(b)** $f(x) = (x-2)^3$

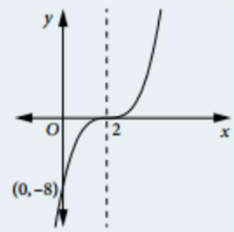
$f(x) = 0$ at $x = 2$

The graph of $f(x) = (x-2)^3$ can be obtained from the graph of $f(x) = x^3$ by a translation of 2 units to the right, parallel to the $x$-axis.

For all $x < 2, f(x) < 0$

For all $x > 2, f(x) > 0$

The function does not pass through the origin, so it cannot be odd.

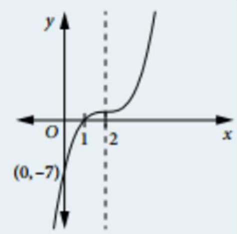



**(c)** $f(x) = (x-2)^3 + 1$

$f(x) = 0$ at $x = 1$

The graph of $f(x) = (x-2)^3 + 1$ can be obtained from the graph of $f(x) = (x-2)^3$ by a translation of 1 unit upwards, parallel to the $y$-axis.

In general, the graph of $f(x) = (x+b)^n + c$ will have the same general shape as the graph of $f(x) = x^n$.



# POLYNOMIAL FUNCTIONS

## Cubic functions

A general cubic function is a polynomial function $f$ of the 3rd degree, defined by $f(x) = ax^3 + bx^2 + cx + d$, where $a, b, c, d$ are constants and $a \neq 0$. Every cubic polynomial has at least one linear factor of the form $(x + \alpha)$, where $\alpha$ is a real number. Factors like this can be found using the factor theorem.

### Example 17

Sketch the graph of $f$, where $f(x) = (x + 1)(x - 2)(x - 3)$.

### Solution

At the points where the graph of $f$ crosses the $x$-axis, $f(x) = 0$:

i.e. $(x + 1)(x - 2)(x - 3) = 0$

$x = -1, 2, 3$

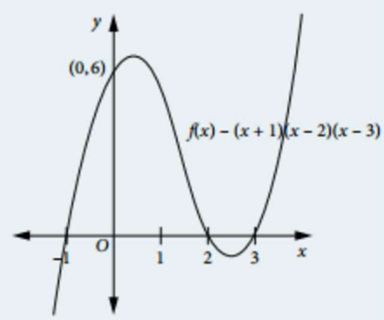
The function value will change sign from positive to negative or from negative to positive at these points ($x = -1$, $x = 2$, $x = 3$).

- For all $x < -1$, each of the three factors is negative, so $f(x) < 0$. e.g. $x = -2$: $f(-2) = -20 < 0$
- For $-1 < x < 2$, the factors $(x + 1) > 0$, $(x - 2) < 0$, $(x - 3) < 0$, so $f(x) > 0$. e.g. $x = 0$: $f(0) = 6 > 0$
- For $2 < x < 3$, the factors $(x + 1) > 0$, $(x - 2) > 0$ and $(x - 3) < 0$, so $f(x) < 0$. e.g. $x = 2.5$: $f(2.5) = -0.875 < 0$
- For all $x > 3$, each of the factors is positive, so $f(x) > 0$. e.g. $x = 4$: $f(4) = 20 > 0$

This information can be summarised in a sign diagram:


At the point where the graph of $f$ crosses the $y$-axis, $x = 0$, so $f(0) = 1 \times (-2) \times (-3) = 6$. Thus the graph crosses the $y$-axis at $(0, 6)$.

*Note:* this cubic equation has a positive coefficient of $x^3$, so that as $x$ increases, $y$ also increases (except in the domain between the two turning points). For $x > 3$, $f(x) > 0$ for all values of $x$.



## Example 18

Sketch the graph of $f$, where $f(x) = 2x^3 - x^2 - 13x - 6$, and find the values of $x$ for which $2x^3 - x^2 - 13x - 6 > 0$.

### Solution

It is useful to find the linear factors of $2x^3 - x^2 - 13x - 6$. (Remember there will always be at least one.)

$f(1) = 2 - 1 - 13 - 6 = -18 \neq 0$ so $(x - 1)$ is not a factor

$f(-1) = -2 - 1 + 13 - 6 = 4 \neq 0$ so $(x + 1)$ is not a factor

$f(2) = 16 - 4 - 26 - 6 = -20 \neq 0$ so $(x - 2)$ is not a factor

$f(-2) = -16 - 4 + 26 - 6 = 0$ so $(x + 2)$ is a factor

To get the other linear factors (if any), you can divide $2x^3 - x^2 - 13x - 6$ by $(x + 2)$ and then factorise the quotient.

$$
\begin{array}{r|l}
 & 2x^2 - 5x - 3 \\
x+2 & 2x^3 - x^2 - 13x - 6 \\
 & \underline{2x^3 + 4x^2} \\
 & \quad -5x^2 - 13x \\
 & \quad \underline{-5x^2 - 10x} \\
 & \qquad -3x - 6 \\
 & \qquad \underline{-3x - 6} \\
 & \qquad\qquad 0
\end{array}
$$

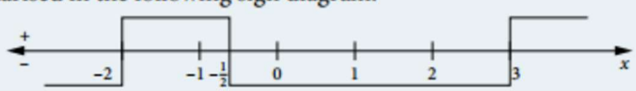
The factors of $2x^2 - 5x - 3$ are $(2x + 1)$ and $(x - 3)$:

$\therefore \quad f(x) = (x + 2)(2x + 1)(x - 3)$

$f(x) = 0$ at $x = -2, -\frac{1}{2}, 3$

- For all $x < -2, f(x) < 0$ e.g. $x = -3$: $f(-3) = -30 < 0$
- For $-2 < x < -\frac{1}{2}, f(x) > 0$ e.g. $x = -1$: $f(-1) = 4 > 0$
- For $-\frac{1}{2} < x < 3, f(x) < 0$ e.g. $x = 0$: $f(0) = -6 < 0$
- For all $x > 3, f(x) > 0$ e.g. $x = 4$: $f(4) = 54 > 0$

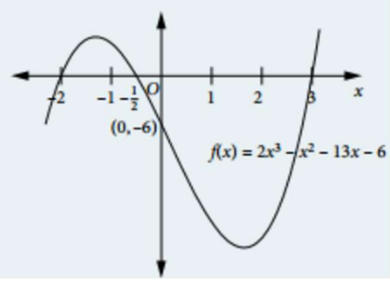
This information is summarised in the following sign diagram.




Hence $f(x) > 0$ for $-2 < x < -\frac{1}{2}$ and for $x > 3$.

At $x = 0, f(0) = -6$, so the graph crosses the $y$-axis at $(0, -6)$.

Note that this cubic equation has a positive coefficient of $x^3$, so that as $x$ increases, $y$ also increases (except in the domain between the two turning points). For $x > 3, f(x) > 0$ for all values of $x$.



### Example 19

Without showing too much detail, sketch graphs of the polynomial functions defined by the following rules.

**(a)** $y = (x+2)(x-1)^2(x+1)$  **(b)** $y = x^2(x-3)(x+1)^3$

### Solution

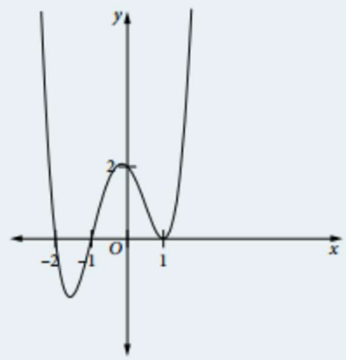
**(a)** $y = (x+2)(x-1)^2(x+1)$. When $|x|$ is large, $y$ behaves like $x^4$.

$(x-1)^2 \geq 0$, so the sign of $y$ is determined by $(x+2)(x+1)$.

Hence $y \geq 0$ for $x < -2$ and for $x > -1$ (equality at $x = 1$);
$y < 0$ for $-2 < x < -1$.

Near $x = -2$, $y$ behaves like a multiple of $(x+2)$ and cuts the $x$-axis like a straight line at $x = -2$. Similarly at $x = -1$.

Near $x = 1$, $y$ behaves like a multiple of $(x-1)^2$, so that $x = 1$ is a stationary point (minimum turning point) and the $x$-axis is a tangent to the curve at $x = 1$.




**(b)** $y = x^2(x-3)(x+1)^3$. When $|x|$ is large, $y$ behaves like $x^5$.

$x^2 \geq 0$, $(x+1)^2 \geq 0$, so the sign of $y$ is determined by $(x-3)(x+1)$.

Hence $y \geq 0$ for $x < -1$ and for $x > 3$ (equality at $x = -1$ and $x = 3$);
$y \leq 0$ for $-1 < x < 3$ (equality at $x = 0$).

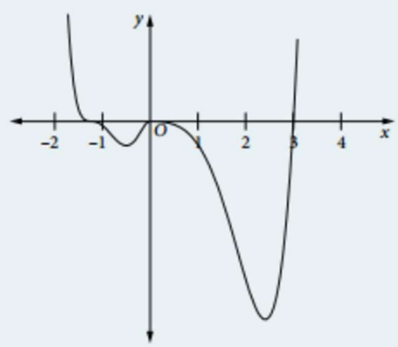
Near $x = 3$, $y$ behaves like a multiple of $(x-3)$ and cuts the $x$-axis like a straight line at $x = 3$.

Near $x = 0$, $y$ behaves like a multiple of $x^2$ and the graph touches the $x$-axis at $x = 0$.

Near $x = -1$, $y$ behaves like a multiple of $(x+1)^3$ and the graph cuts the $x$-axis at $x = -1$ like a basic cubic curve, i.e. with a horizontal point of inflection.

This sketch is **not** drawn to scale. The minimum between $x = -1$ and $x = 0$ cannot be shown at a reasonable scale without the other minimum being below the page.

You can use graphing software to draw this graph. As you zoom out to find the minimum turning point between $x = 0$ and $x = 3$, the curve between $x = -1$ and $x = 0$ appears to become a straight line.




## Summary of polynomial functions

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, $a_n \neq 0$, then:

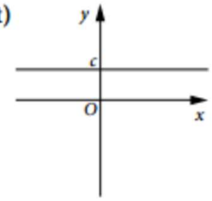
1. For very large $|x|$, $P(x) \approx a_n x^n$.
2. A polynomial of odd degree always has at least one real zero (i.e. its graph cuts the $x$-axis at least once).
3. At least one maximum or minimum value of $P$ occurs between any two distinct real zeros.
4. For a polynomial of odd degree, the ends of the graph go in opposite directions.
5. For a polynomial of even degree, the ends of the graph go in the same direction.
6. When the graph of a polynomial function meets the $x$-axis, it may cut it (single zero), touch it (double zero) or cut it at a point of inflection (triple zero).

# POLYNOMIAL FUNCTIONS

## Polynomial graphs, standard forms

**Degree 0 (constant)**

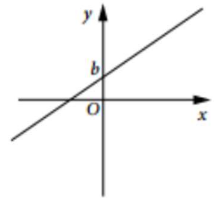
$P(x) = c$




Does not cut
No zeros

**Degree 1 (linear)**

$P(x) = mx + b$

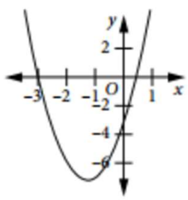



Cuts once
One zero

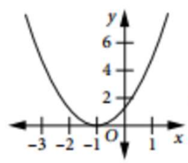
**Degree 2 (quadratic)**

$P(x) = ax^2 + bx + c$

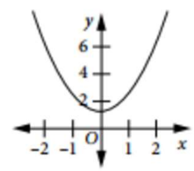
**$a > 0$:**



Cuts twice
$\Delta > 0$
Two zeros

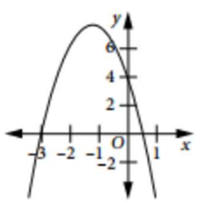


Touches
$\Delta = 0$
Double zero

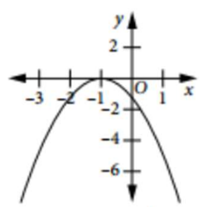


Does not cut
$\Delta < 0$
No real zeros

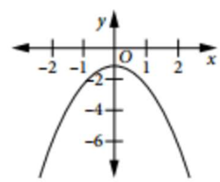
**$a < 0$:**



Cuts twice
$\Delta > 0$
Two zeros



Touches
$\Delta = 0$
Double zero



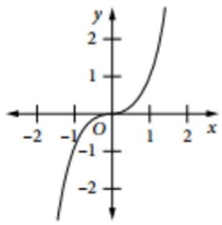
Does not cut
$\Delta < 0$
No real zeros

A quadratic polynomial may have two, one or no real zeros.

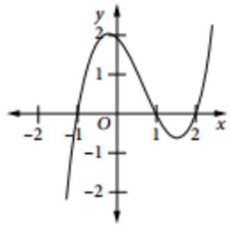
**Degree 3 (cubic)**

$P(x) = ax^3 + bx^2 + cx + d$

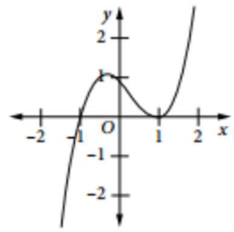
$\boldsymbol{a > 0}$:



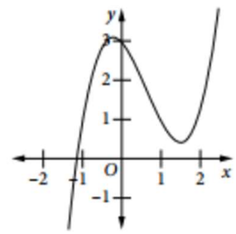

Cuts once
Triple zero




Cuts three times
Three zeros

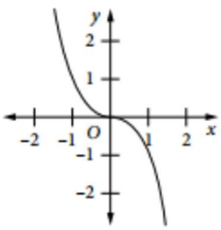



Cuts once and touches
One zero, one double zero

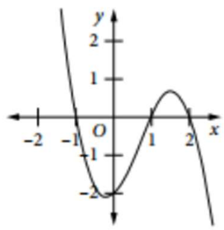



Cuts once
One zero

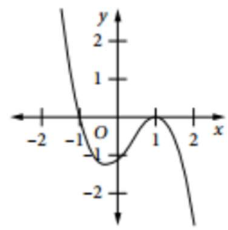
$\boldsymbol{a < 0}$:



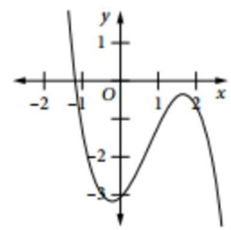

Cuts once
Triple zero




Cuts three times
Three zeros




Cuts once and touches
One zero, one double zero




Cuts once
One zero

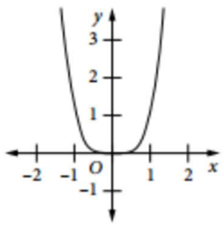
A cubic polynomial may have one, two or three real zeros.

A cubic polynomial always has at least one real zero.

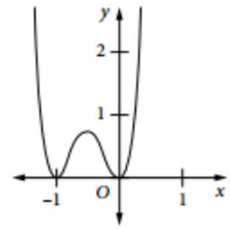
**Degree 4 (quartic)**

$P(x) = ax^4 + bx^3 + cx^2 + dx + e$

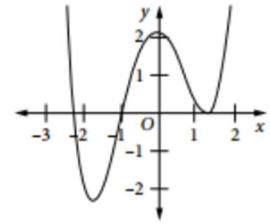
$\boldsymbol{a > 0}$**:**



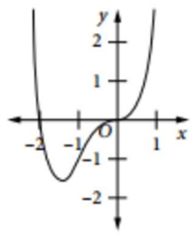

Touches
Quadruple zero



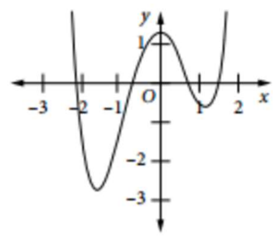

Touches twice
Two double zeros



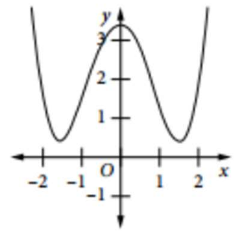

Touches once, cuts twice
Two zeros, one double zero




Cuts twice
One zero, one triple zero




Cuts four times
Four zeros




Does not cut
No real zeros

The addition of an appropriate constant to each equation can create a polynomial with no real zeros.

$\boldsymbol{a < 0}$**:**

A negative $a$ inverts each of the six graphs above so that they open downwards, with the properties of their zeros the same.

## Summary of quartic polynomials

1. A quartic polynomial may have four, three, two, one or no real zeros.
2. If a quartic polynomial has only one real zero, then it must be a quadruple zero.
3. If a quartic polynomial has only two distinct real zeros, then they are either a triple zero and a single zero or they are both double zeros.
4. If a quartic polynomial has only three distinct real zeros, then they are a double zero and two single zeros.
5. If a quartic polynomial has four distinct real zeros, then it can be factorised into four real linear factors.
6. If a quartic polynomial has no real zeros, then it cannot be factorised into any real linear factors.