BASICS — PARABOLAS of the form $y = ax^2 + bx + c$

Equations of the form

$$y = ax^2 + bx + c$$

represent parabolas.

Domain : all real x

Range : restricted
(look for y-coordinate of vertex).

Graphing Parabolas

When graphing parabolas:

- Label the axes, origin, and equation of each curve.
- you must show the x- and y- intercepts and the VERTEX

Features of $y = ax^2 + bx + c$

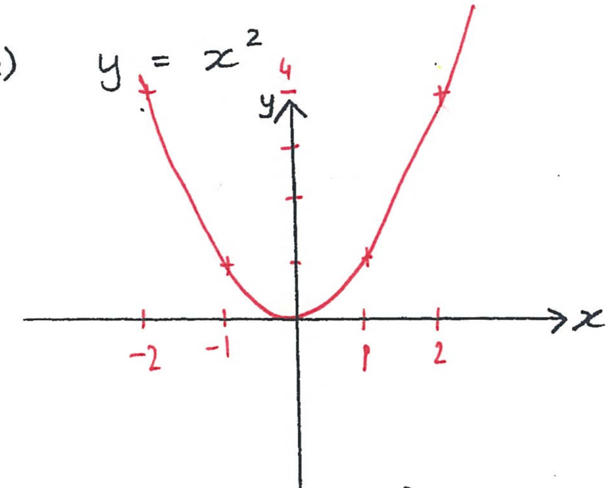
- If $a > 0$, the parabola is concave up
- If $a < 0$, the parabola is concave down
- The larger the value of $|a|$, the narrower the parabola.
- Equation of the axis of symmetry is:

$$x = \frac{-b}{2a}$$

Examples: The BASICS

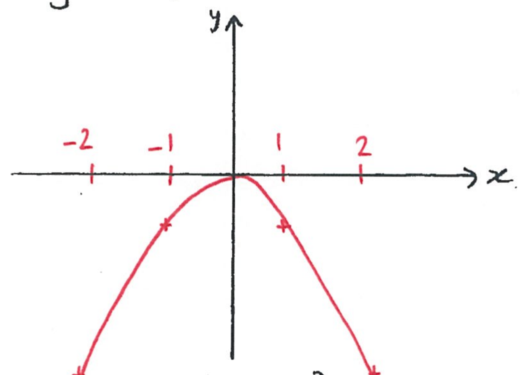
① Concavity

a) $y = x^2$



Function or relation?
Domain :
Range :

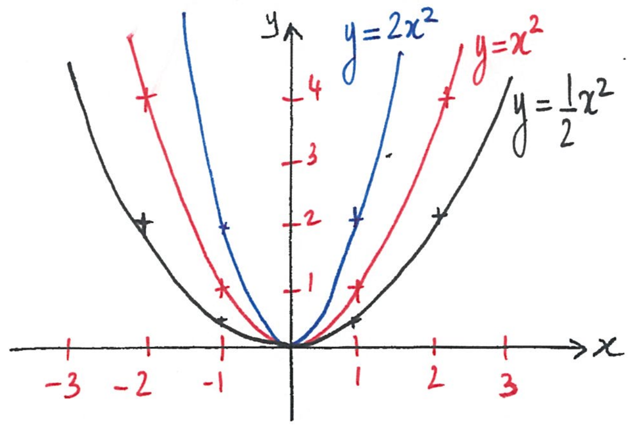
b) $y = -x^2$



Function or relation?
Domain :
Range :

② Curvature

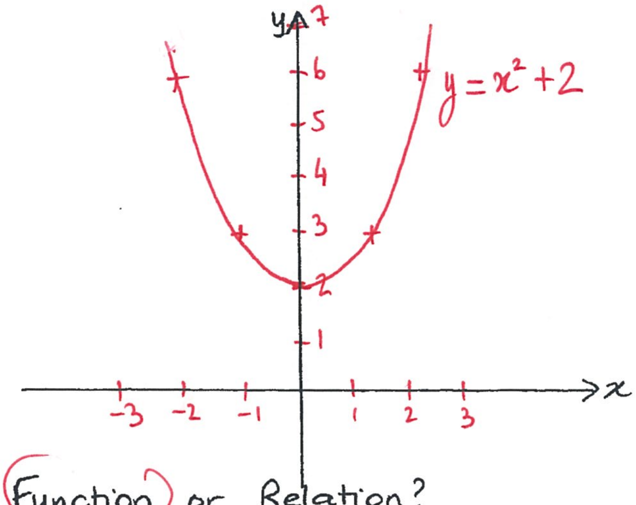
$y = x^2$, $y = 2x^2$, $y = \frac{1}{2}x^2$



③ Shifting Up/Down y-axis

For $y = ax^2 + c$, the value of c determines the y-intercept.

a) $y = x^2 + 2$

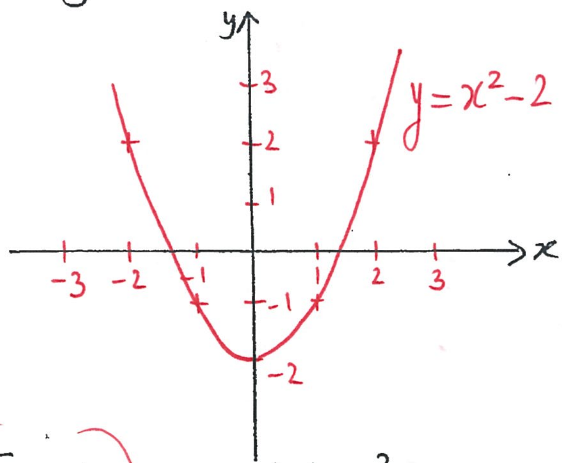


Function or Relation?

Domain: $\mathbb{R}$

Range: $[2, +\infty)$

b) $y = x^2 - 2$

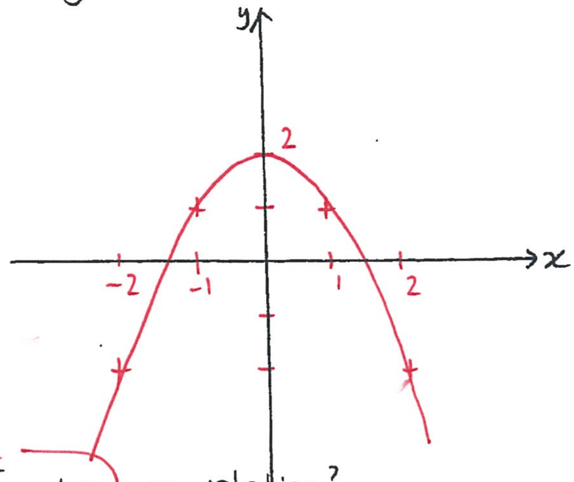


Function or relation?

Domain: $\mathbb{R}$

Range: $[-2, +\infty)$

c) $y = 2 - x^2$

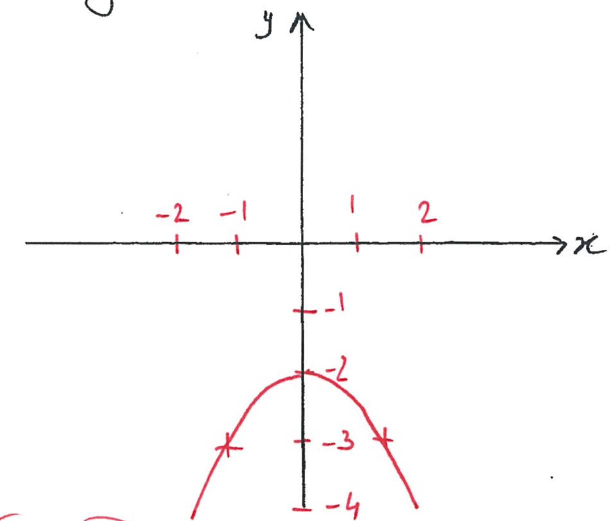


Function or relation?

Domain: $\mathbb{R}$

Range: $(-\infty, 2]$

d) $y = -x^2 - 2$



Function or relation?

Domain: $\mathbb{R}$

Range: $(-\infty, -2]$

BASICS — PARABOLAS OF THE FORM $y = ax^2 + bx + c$

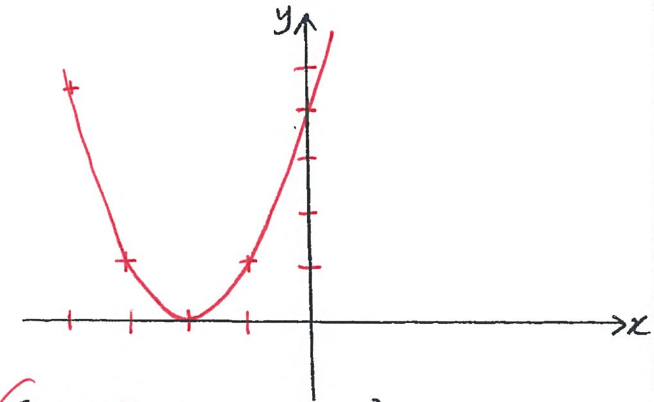
④ Shifting along the x-axis

If k is a constant,

$y = (x+k)^2 \Rightarrow$ shift to LEFT

$y = (x-k)^2 \Rightarrow$ shift to RIGHT

a) $y = (x+2)^2$



Function or relation?

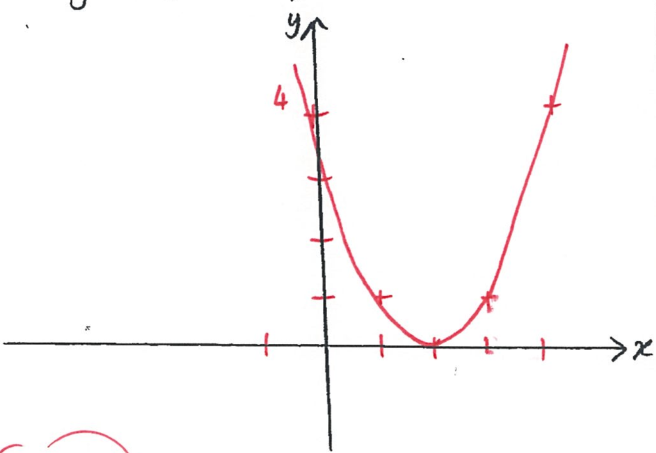
Domain: $\mathbb{R}$

Range: $\mathbb{R}^+$

Axis of Symmetry: $x = -2$

Vertex: $(-2, 0)$

b) $y = (x-2)^2$



Function or relation?

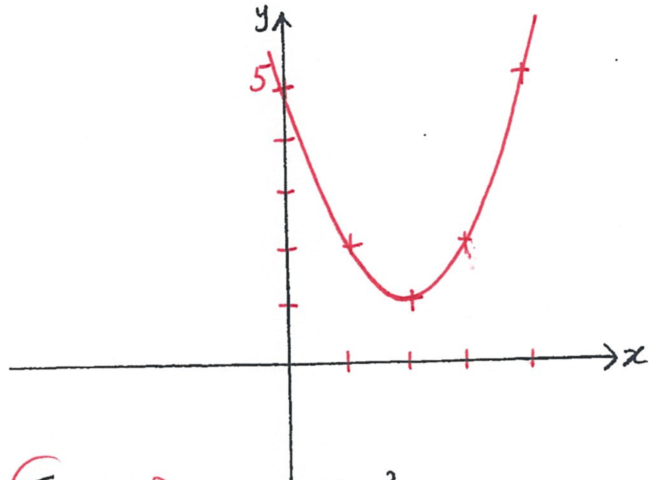
Domain: $\mathbb{R}$

Range: $\mathbb{R}^+$

Axis of Symmetry: $x = 2$

Vertex: $(2, 0)$

c) $y = (x-2)^2 + 1$



Function or relation?

Domain: $\mathbb{R}$

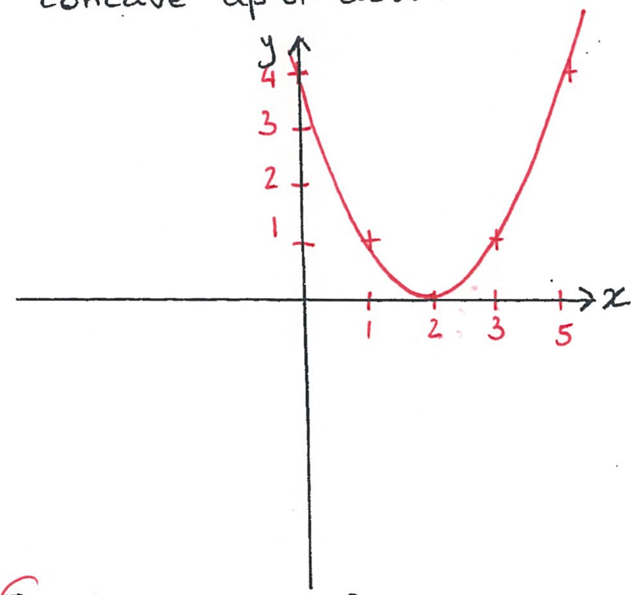
Range: $[1, +\infty)$

Axis of Symmetry: $x = 2$

Vertex: $(2, 1)$

d) $y = (2-x)^2$

concave up or down?



Function or relation?

Domain: $\mathbb{R}$

Range: $\mathbb{R}^+$

Axis of Symmetry: $x = 2$

Vertex: $(2, 0)$

PARABOLAS OF THE FORM $y=ax^2+bx+c$

Graphing more complex parabolas

To graph more complex parabolas, follow these steps:

1. Determine if the curve is concave up or down.
2. Factorise the equation of the curve. (if possible)
3. Find any x-intercepts, by solving $y=0$.
4. Find the axis of symmetry.
   * By inspection: halfway between the x-values obtained in (3).
   * By formula: $x=\frac{-b}{2a}$
5. Find the coordinates of the vertex. (Vertex lies on axis of symmetry).
6. Find y-intercept, by letting $x=0$.

Note: [A] If the equation of the curve can not be easily factorised to find the solutions when $y=0$, use the quadratic formula instead

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

[B] If there are no solutions when $y=0$, then there are no x-intercepts, and you must use the axis of symmetry, vertex and y-intercept to draw the graph. (see page 11)

Examples: Sketch:

① $y=x^2-4x$

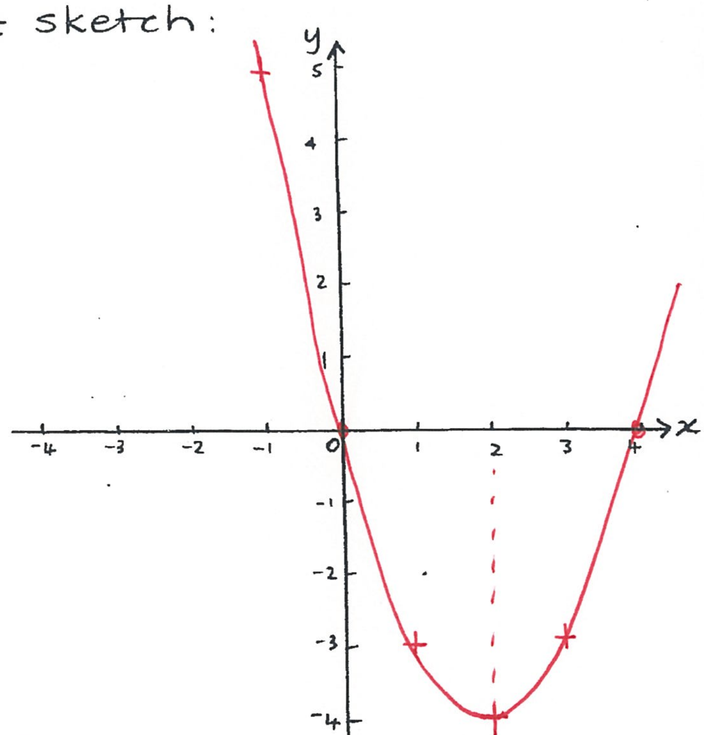
* concave up/down? up
* factorise: $y=x(x-4)$
* x-intercepts (solve for $y=0$)

$y=0$ when $x=0$ or $x=4$

* axis of symmetry:

$x=2$

* vertex: $(2,-4)$
* y-intercept: $y=0$
* sketch:



Domain:

Range:

PARABOLAS OF THE FORM $y=ax^2+bx+c$

② $y = x^2 - 2x - 3$

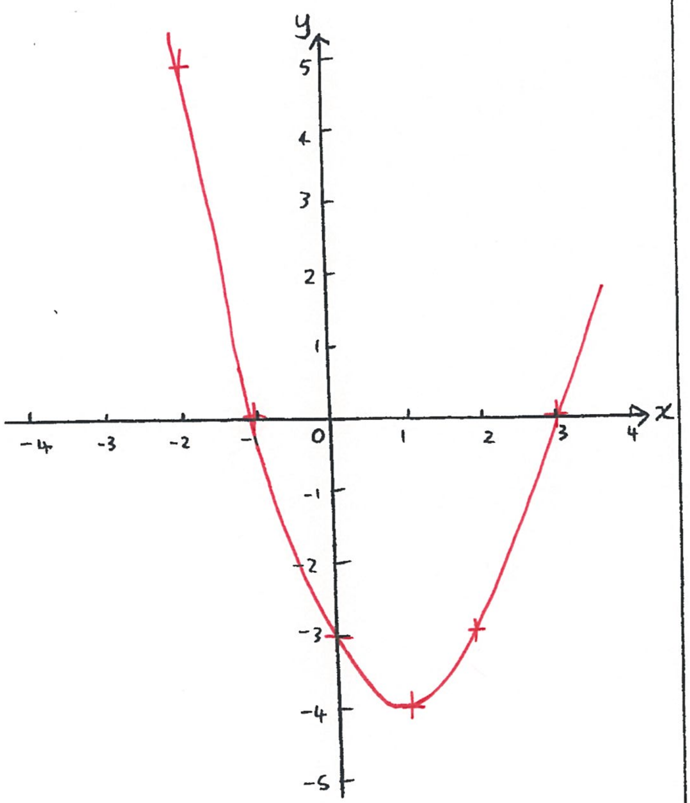
* concave up
* factorise $(x+1)(x-3)$
* x-intercepts (solve for $y=0$)

$y=0$ when $x=-1$ or $x=3$

* axis of symmetry

$x=1$

* vertex $(1,-4)$
* y-intercept $y=-3$



Domain: $\mathbb{R}$

Range: $[-4,+\infty)$

③ $y = x^2 - 6x + 9$

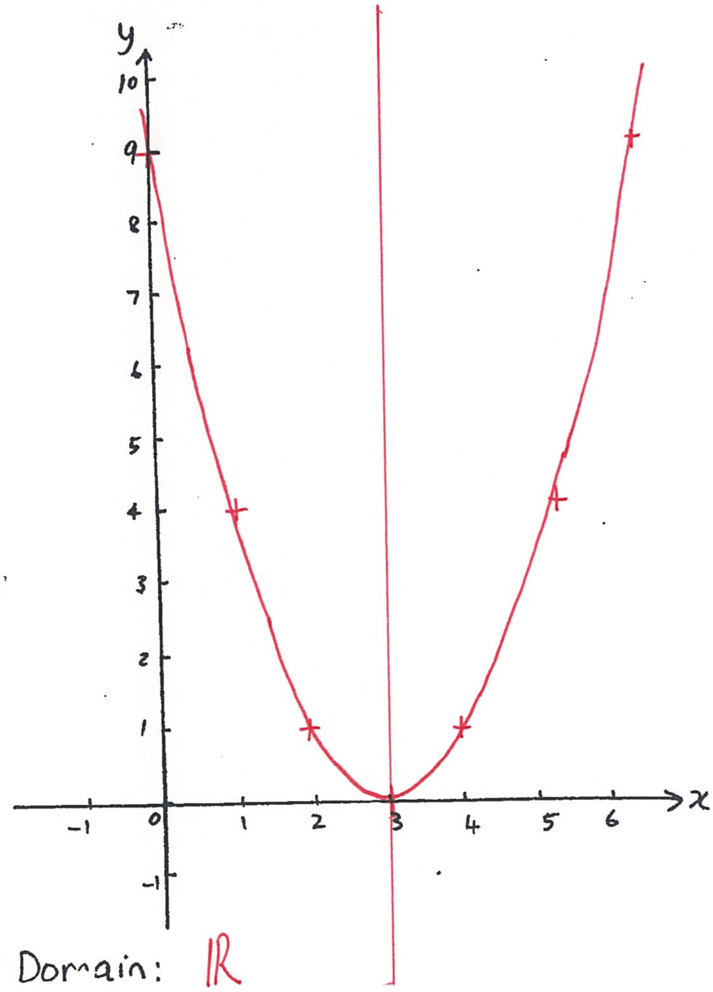
* concave: up
* factorise: $(x-3)(x-3)$
* x-intercepts (solve for $y=0$)

$x=3$

* axis of symmetry:

$x=3$

* vertex $(3,0)$
* y-intercept



Domain: $\mathbb{R}$

Range: $\mathbb{R}^+$

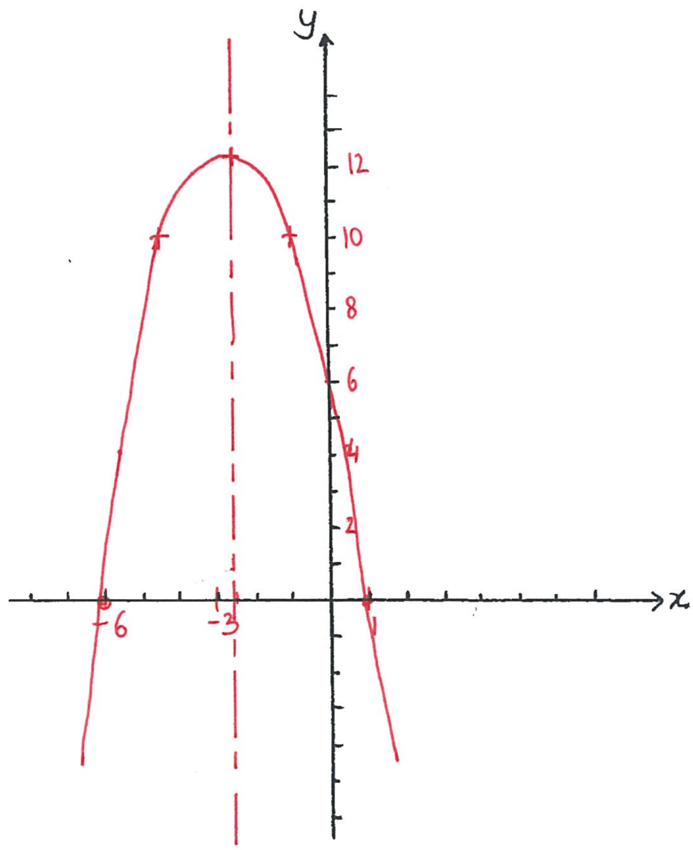
PARABOLAS OF THE FORM $y = ax^2 + bx + c$

④ $y = 6 - 5x - x^2$

* concave down
* factorise $(x-1)(-x-6)$
* x-intercepts (solve for $y=0$)

  $x = -6 \quad x = 1$
* axis of symmetry

  $x = -\frac{5}{2}$
* vertex $\left(-\frac{5}{2}, 12.25\right)$
* y-intercept $y = 6$



Domain: $\mathbb{R}$

Range: $(-\infty, 12.25]$

⑤ $y = 9 - x^2$

concave down

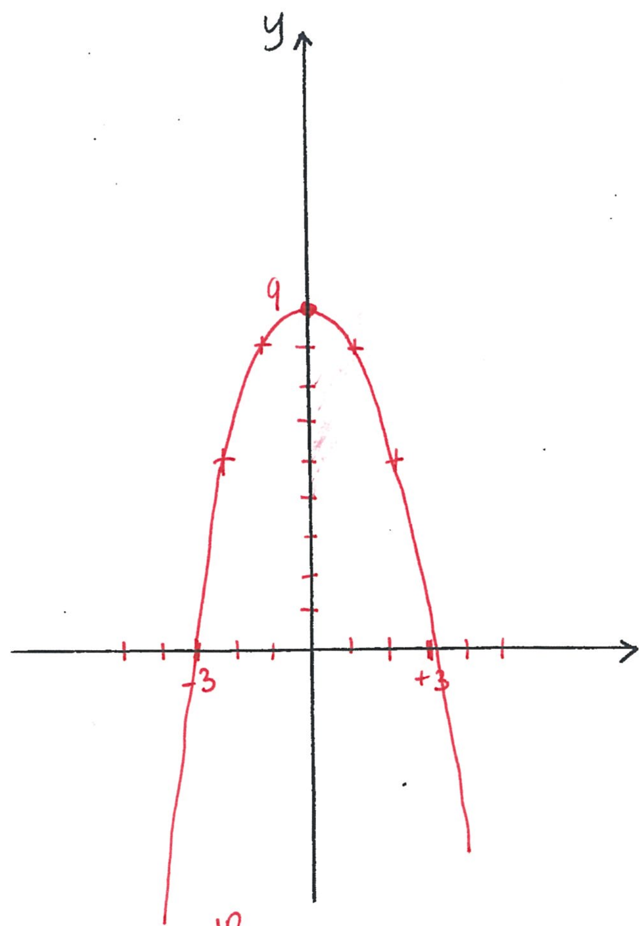
$y = (3-x)(3+x)$

x intercepts are (-3) and $(+3)$

axis of symmetry $x = 0$

vertex $(0, 9)$

y intercept $~~x~~\, y = 9$



Domain: $\mathbb{R}$

Range: $(-\infty, 9]$

⑥ $y = x^2 - 4x + 5$

concave up

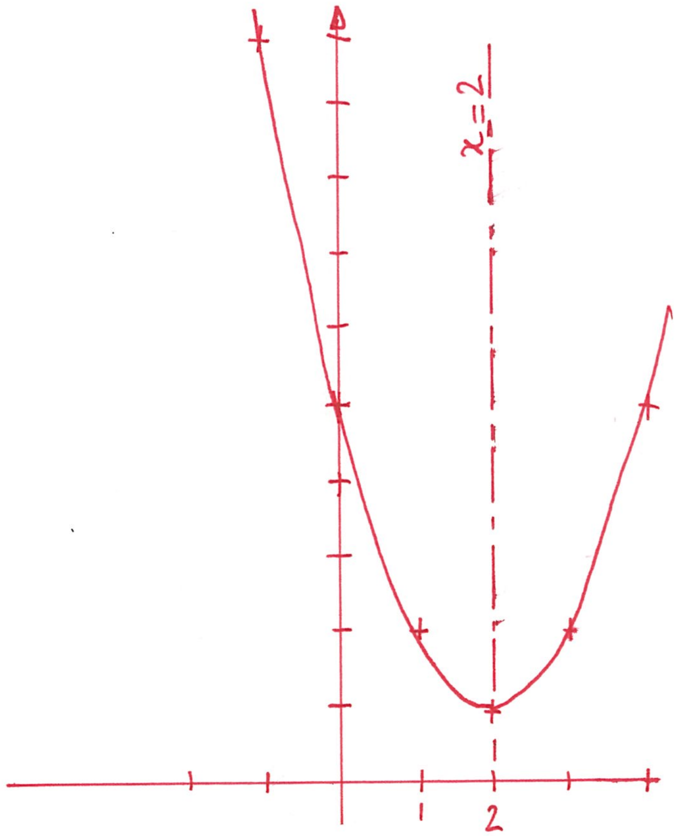
$\Delta = 16 - 4 \times 5 = -4 < 0$

so no roots.

Axis of symmetry $x = -\frac{b}{2a} = \frac{4}{2} = 2$

For $x = 2$, $y = 4 - 4 \times 2 + 5 = 1$

So vertex is $(2, 1)$



Minimum & Maximum Values of Parabolas

▶ Minimum values

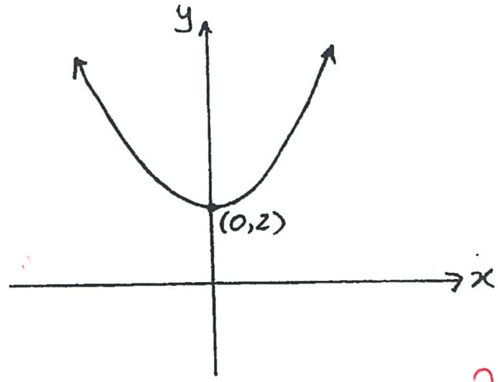
If the coefficient of x^2 is positive, the graph is concave upwards, and it has a minimum value.

This minimum value is the y-coordinate of the vertex.

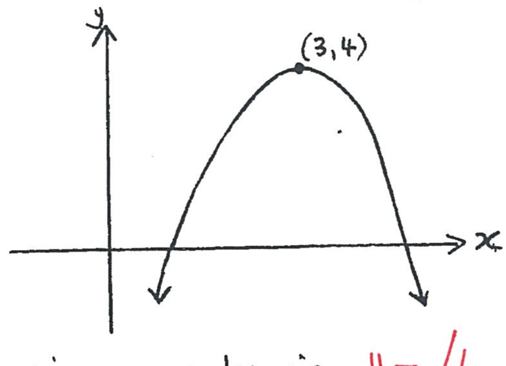
▶ Maximum values

If the coefficient of x^2 is negative, the graph is concave downwards, and it has a maximum value.

This maximum value is the y-coordinate of the vertex.

① 

Minimum value is $y = 2$

② 

maximum value is $y = 4$

QUADRATIC FUNCTIONS (Parabolas)

Quadratic functions are of the form: $f(x) = ax^2 + bx + c$ where a, b, c are constants; this is called the "**general form**" of a quadratic function. These curves are called "**parabolas**". If $a > 0$, the parabola is **concave up**; if $a < 0$, the parabola is **concave down.**

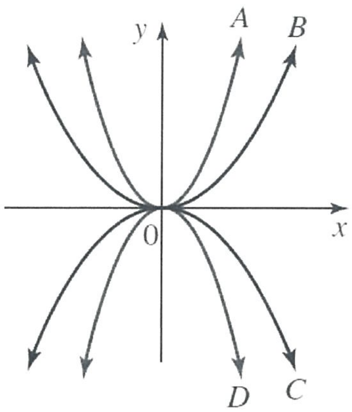
In fact, the general form of a parabola (i.e. $f(x) = ax^2 + bx + c$, as noted above) can be rearranged as two possible other forms:

1) $f(x) = a(x-h)^2 + k$ this form is called the "**vertex form**" as (h, k) are the coordinates of the vertex of the parabola.

2) $f(x) = a(x-r_1)(x-r_2)$ this form is called the "**intercept form**" as $(r_1, 0)$ and $(r_2, 0)$ are the coordinates of the x-intercepts of the parabola.

When asked to find the equation of a parabola from a graph, we use one of these 3 forms, depending on the information provided by the graph.

10



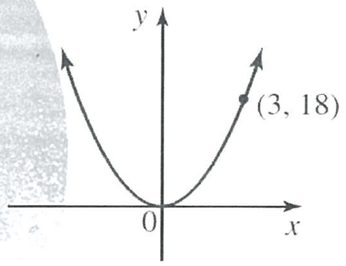
Four parabolas $y = x^2$, $y = -x^2$, $y = 2x^2$ and $y = -2x^2$ have been drawn on the same number plane. State the graph whose equation is:

a $y = x^2$ B **b** $y = -x^2$ C

c $y = 2x^2$ A **d** $y = -2x^2$ D

11 The curves below are parabolas with equations of the form $y = ax^2$, where a is a constant. For each curve, find the value of a and hence determine its equation.

a



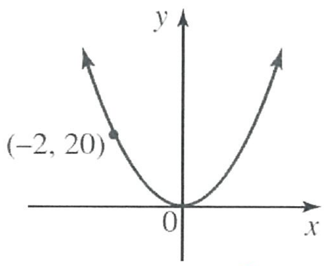
$y = ax^2$

$18 = a \times 3^2$

so $a = 2$

$y = 2x^2$

b

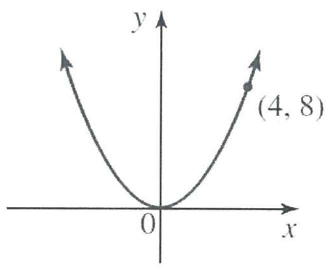


$20 = a(-2)^2 = 4a$

so $a = 5$

$y = 5x^2$

c



$8 = a \times 4^2$

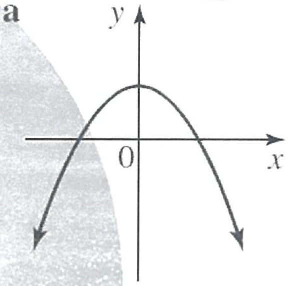
$a = \frac{8}{16} = \frac{1}{2}$

$y = \frac{1}{2}x^2$

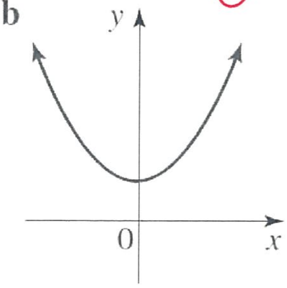
6 Match each of these equations with one of the graphs below.

- $y = x^2 + 3$ b
- $y = x^2 - 3$ d
- $y = 3 - x^2$ a
- $y = -x^2 - 3$ d

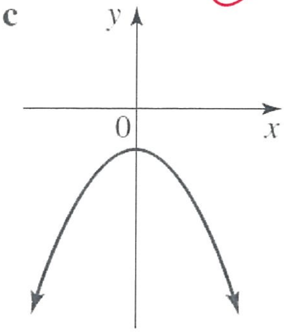
a



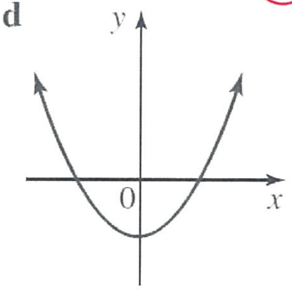
b



c



d



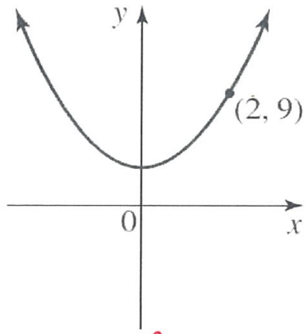
7 Find the equation of the new parabola if the curve $y = x^2 + 2$ is translated:

a 6 units up **b** 2 units down **c** 5 units down

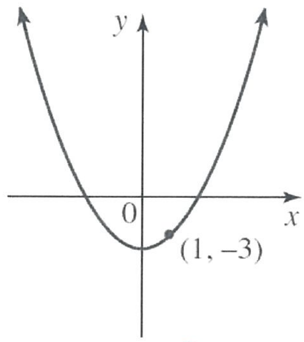
$y = x^2 + 8$ $y = x^2$ $y = x^2 - 3$

10 The curves below are parabolas with equations of the form $y = x^2 + c$ or $y = -x^2 + c$. For each curve, find the value of c and hence determine its equation.

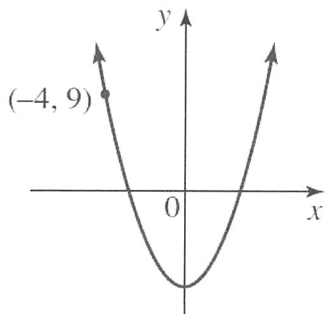
a



b



c



$9 = 2^2 + c$

so $c = 5$

$y = x^2 + 5$

$-3 = 1^2 + c$

so $c = -4$

$y = x^2 - 4$

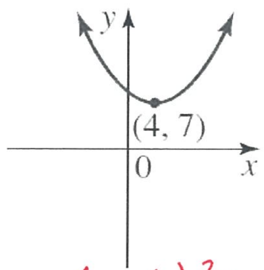
$9 = (-4)^2 + c$

so $c = 9 - 16 = -7$

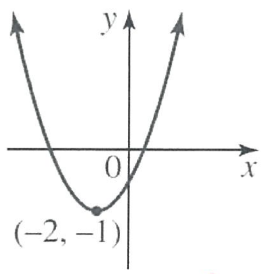
$y = x^2 - 7$

9 Find the equation of each curve in the form $y = a(x - h)^2 + k$, where $a = 1$ or -1.

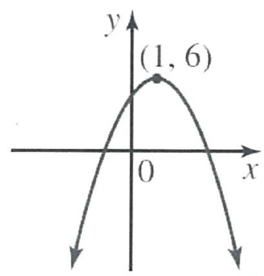
a



b



c



$y = (x - h)^2 + k$

$y = (x - 4)^2 + 7$

$y = (x - h)^2 + k$

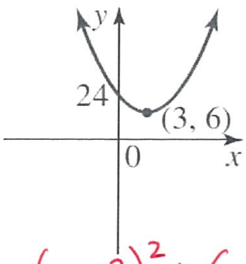
$y = (x + 2)^2 - 1$

$y = -(x - h)^2 + k$

$y = -(x - 1)^2 + 6$

10 Find the equation of each parabola in the form $y = a(x-h)^2 + k$.

a



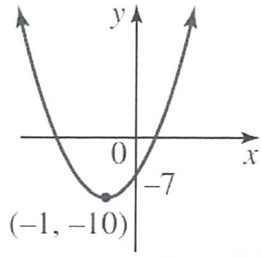
$y = a(x-3)^2 + 6$

But $(0,24)$ belongs to the parabola

thus: $24 = 9a + 6$ so $a = 2$

$y = 2(x-3)^2 + 6$

b



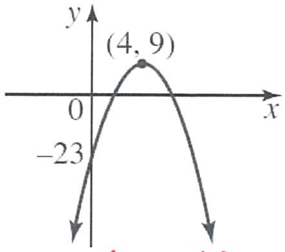
$y = a(x+1)^2 - 10$

$-7 = a \times 1^2 - 10$

$a = 3$

$y = 3(x+1)^2 - 10$

c



$y = a(x-4)^2 + 9$

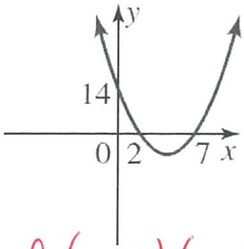
$-23 = a \times (-4)^2 + 9$

$\therefore a = \frac{-32}{16} = -2$

$y = -2(x-4)^2 + 9$

10 Find the equation of each parabola in the form $y = k(x-a)(x-b)$.

a



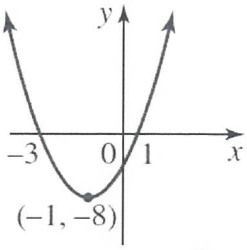
$y = k(x-2)(x-7)$

$14 = k(-2)(-7)$

$\therefore k = 1$

$y = (x-2)(x-7)$

b



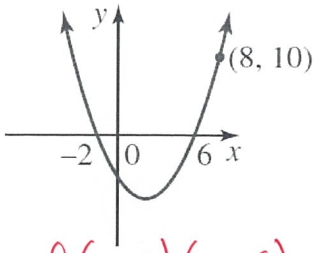
$y = k(x+3)(x-1)$

$-8 = k(2)(-2)$

so $k = 2$

$y = 2(\cancel{x}+3)(x-1)$

c



$y = k(x+2)(x-6)$

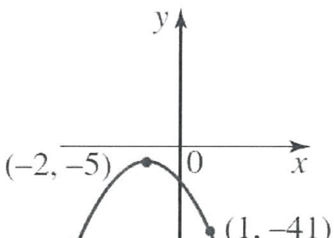
$10 = k(10) \times 2$

so $k = \frac{1}{2}$

$y = \frac{1}{2}(x+2)(x-6)$

Find the equation of each parabola, using either the standard, vertex or intercept forms.

d



$y = a(x-h)^2 + k$

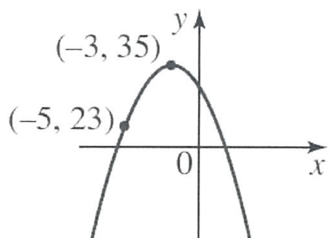
$y = a(x+2)^2 - 5$

then $-41 = a(1+2)^2 - 5$

$a = \frac{-36}{9} = -4$

$y = -4(x+2)^2 - 5$

e



$y = a(x+3)^2 + 35$

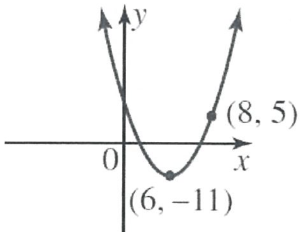
then $23 = a(-5+3)^2 + 35$

$23 = 4a + 35$

so $a = -3$

$y = -3(x+3)^2 + 35$

f



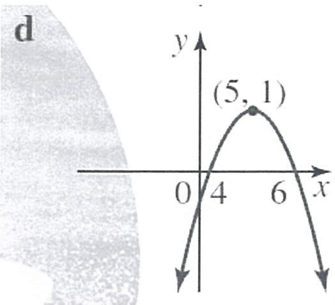
$y = a(x-6)^2 - 11$

$5 = a(8-6)^2 - 11$

$\therefore a = \frac{16}{4} = 4$

So $y = 4(x-6)^2 - 11$

d



$y = a(x-5)^2 + 1$

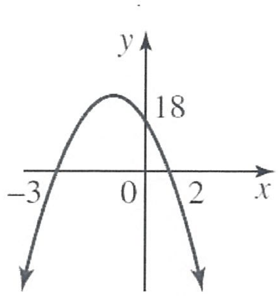
or $y = a(x-4)(x-6)$

$1 = a(5-4)(5-6) = -a$

so $a = -1$

$y = -(x-4)(x-6)$

e



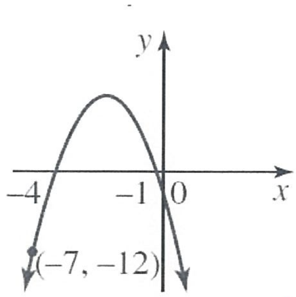
$y = a(x+3)(x-2)$

$18 = a \times 3 \times (-2)$

so $a = -3$

$y = -3(x+3)(x-2)$

f



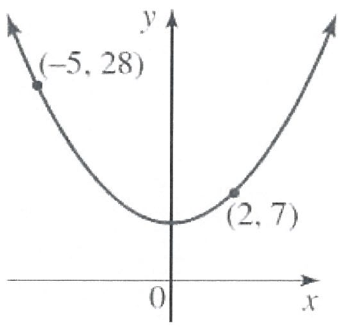
$y = a(x+4)(x+1)$

$-12 = a(-7+4)(-7+1)$

so $a = \frac{-12}{(-3)\times(-6)} = \frac{-2}{3}$

$y = -\frac{2}{3}(x+4)(x+1)$

12



The parabola shown has an equation of the form $y = ax^2 + c$. Form a pair of simultaneous equations and hence find the equation of the parabola.

$$\begin{cases} 28 = a(-5)^2 + c = 25a + c & \text{equation (1)} \\ 7 = a \times 2^2 + c = 4a + c & \text{equation (2)} \end{cases}$$

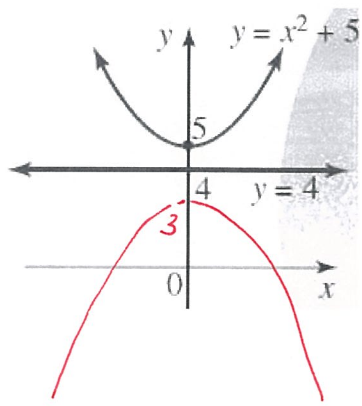
By elimination: equation (1) − equation (2) gives: $21 = 21a$ so $a = 1$

And then, substituting this value in any of the equations,

we obtain: $7 = 4 \times 1 + c$ so $c = 3$

$y = x^2 + 3$

13 What would be the equation of the new parabola if the curve $y = x^2 + 5$ is reflected in the line $y = 4$?



$y = -x^2 + 3$

11 Find the equation of the parabola that passes through the points (0, 4), (1, 5) and (−3, 25).
[Hint: Start $y = ax^2 + bx + c$ and find c first.]

$$\begin{cases} 4 = a\times 0^2 + b\times 0 + c & \text{so } c = 4 \\ 5 = a\times 1^2 + b\times 1 + 4 = a + b + 4 \\ 25 = a\times(-3)^2 + b\times(-3) + 4 = 9a - 3b + 4 \end{cases}$$

so $\begin{cases} a + b = 1 & (1) \\ 9a - 3b = 21 & (2) \end{cases}$

(2) + 3×(1) gives $12a = 21 + 3 = 24$ so $a = 2$

and then $b = 1 - a = 1 - 2 = -1$

$$y = 2x^2 - x + 4$$

Transform the equation of the parabola from standard form to vertex form by completing the square.

a) $y = 2x^2 - 4x + 5$

$y = 2(x^2 - 2x) + 5$

$y = 2[(x-1)^2 - 1] + 5$

$y = 2(x-1)^2 - 2 + 5$

$y = 2(x-1)^2 + 3$

Vertex is $(1,3)$

Concave up

y intercept is $y = 5$

b) $y = -3x^2 - 12x - 7$

$y = -3(x^2 + 4x) - 7$

$y = -3[(x+2)^2 - 4] - 7$

$y = -3(x+2)^2 + 12 - 7$

$y = -3(x+2)^2 + 5$

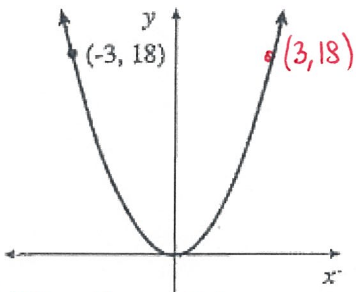
Vertex is $(-2,5)$

Concave down

y intercept is $y = -7$

Find the equation of each parabola

1



$y = ax^2 + bx + c$

but $c = 0$

$y = ax^2 + bx$

$y = x(ax + b)$

$\begin{cases} 18 = 9a - 3b \\ 18 = 9a + 3b \end{cases}$ so $18a = 36$ $a = 2$

and $b = 0$

$y = 2x^2$

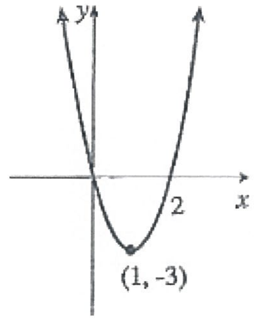
$y = k(x-1)^2 - 3$

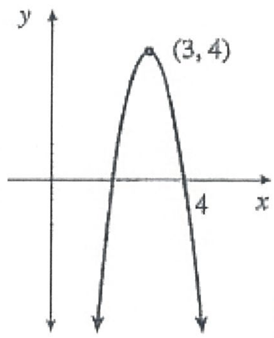
or $y = a(x-2)x$

$-3 = a(1-2)$

so $a = 3$

$y = 3x(x-2)$





$y = k(x-3)^2 + 4$

$0 = k(4-3)^2 + 4$

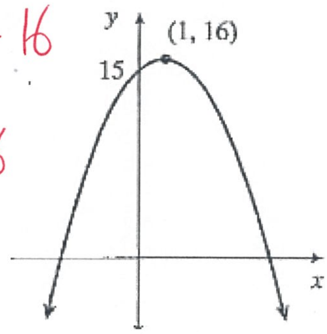
so $k = -4$

$y = -4(x-3)^2 + 4$

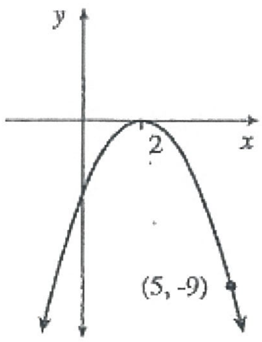
$y = k(x-1)^2 + 16$

$15 = k(-1)^2 + 16$

so $k = -1$



$y = -(x-1)^2 + 16$



$y = k(x-2)^2$

$-9 = k(5-2)^2$

so $k = -1$

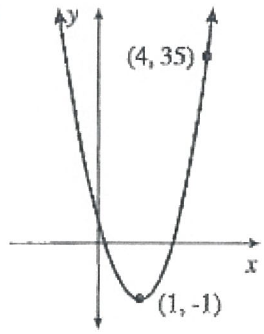
$y = -(x-2)^2$

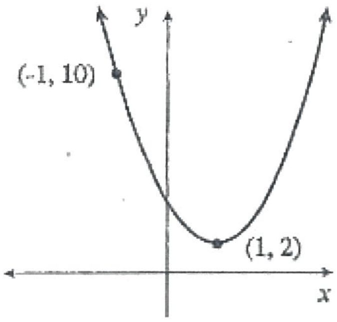
$y = k(x-1)^2 - 1$

$35 = k(4-1)^2 - 1$

$k = \frac{36}{3^2} = 4$

$y = 4(x-1)^2 - 1$





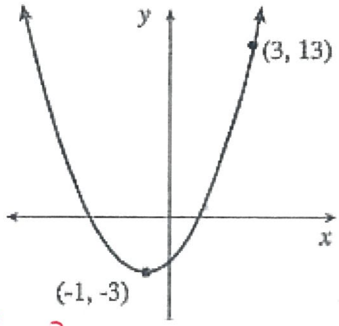
$y = k(x-1)^2 + 2$

$10 = k(-1-1)^2 + 2$

$10 = 4k + 2$

$k = 2$

$y = 2(x-1)^2 + 2$

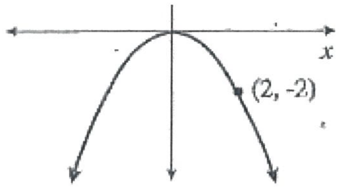


$y = k(x+1)^2 - 3$

$13 = k(3+1)^2 - 3$

$k = \frac{16}{4^2} = 1$

$y = (x+1)^2 - 3$



$y = ax^2 + 0$

$-2 = a \times 2^2$ so $a = -\frac{1}{2}$

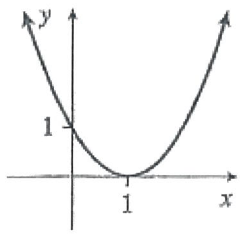
$y = -\frac{1}{2}x^2$

$y = k(x-1)^2$

$1 = k(0-1)^2$

so $k = 1$

$y = (x-1)^2$



11 A member of an indoor cricket team, playing a match in a gymnasium, hits a ball that follows a path given by $y = -0.1x^2 + 2x + 1$, where y is the height above ground, in metres, and x is the horizontal distance travelled by the ball.

The ceiling of the gymnasium is 10.6 metres high. Will this ball hit the roof? Explain.

$y = -0.1x^2 + 2x + 1 \iff y = -0.1\left[x^2 - \frac{2}{0.1}x\right] + 1$

$\iff y = -0.1\left[x^2 - 20x\right] + 1 = -0.1\left[(x-10)^2 - 100\right] + 1$

$y = -0.1(x-10)^2 + 10 + 1$

So $y = -0.1(x-10)^2 + 11$ vertex is $(10, 11)$

$11 > 10.6$ so the ball will hit the roof by 40 cm.