ARITHMETIC SEQUENCES

SEQUENCES

The word **sequence** is often used in everyday language. You say 'a sequence of events'; you complete tasks in a certain sequence; the order of chapters in this book is the sequence in which the Mathematics syllabus is treated.

Whenever the word 'sequence' is used, you are considering a set of objects, ideas, steps or events in some definite order. This order can be associated with the set of positive integers $\{1, 2, 3, \ldots\}$, as is indicated when you use the terms first, second, third etc. to describe a position in the sequence of events.

An **arithmetic sequence** is a sequence of terms in which each term after **the first term** is formed by adding a constant number, called the **common difference**, to the preceding term.

For arithmetic sequences: a is the first term, d is the common difference and n is the number of terms.

The sequence $a, a + d, a + 2d, \ldots, a + (n - 1)d$ is an arithmetic sequence with n terms, where a and d are real numbers and n is a positive integer. The value of each term can be labelled as T_n and defined as follows:

$$T_n = T_{n-1} + d$$

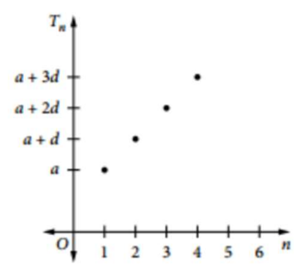
This definition is called a **recursive definition** because it defines each term of the sequence as a function of the preceding term.

Rewriting this definition gives $d = T_n - T_{n-1}$ and this result can be used to identify whether a sequence is arithmetic. If there is a constant difference between all successive terms then the sequence is arithmetic.

An arithmetic sequence may be defined by its nth term, given by:

$$T_n = a + (n - 1)d$$

Graphically, T_n as a function of n is linear, consisting of a series of points for integer values of n, $n > 0$, which follow the straight line given by $T_n = a - d + nd$. The independent variable is n and the dependent variable is T_n.



Some examples of arithmetic sequences are:

$2, 5, 8, 11, \ldots$	$a = 2, d = 3$
$12, 5, -2, -9, \ldots$	$a = 12, d = -7$
$\pi + 1, 2\pi + 4, 3\pi + 7, 4\pi + 10, \ldots$	$a = \pi + 1, d = \pi + 3$
$2 + \sqrt{3}, 2 + 2\sqrt{3}, 2 + 3\sqrt{3}, 2 + 4\sqrt{3}, \ldots$	$a = 2 + \sqrt{3}, d = \sqrt{3}$

Example 9

By finding the difference between successive terms, identify the sequences that are arithmetic in the following:

(a) $11, 15, 19, 23, 27, \ldots$ **(b)** $10, 12, 15, 19, 24, \ldots$ **(c)** $22, 16, 10, 4, -2, \ldots$

(d) $8, 6, 8, 6, 8, \ldots$ **(e)** $1\frac{1}{2}, 2\frac{1}{4}, 3, 3\frac{3}{4}, 4\frac{1}{2}, \ldots$

Solution

(a) $15 - 11 = 19 - 15 = 23 - 19 = 27 - 23 = 4$, a constant. The sequence is arithmetic.

(b) $12 - 10 = 2$, $15 - 12 = 3$, $19 - 15 = 4$, $24 - 19 = 5$. The difference between successive terms is not constant. The sequence is not arithmetic, even though it follows a pattern.

(c) $16 - 22 = 10 - 16 = 4 - 10 = -2 - 4 = -6$, a constant. The sequence is arithmetic.

(d) $6 - 8 = -2$, $8 - 6 = 2$. The difference between successive terms is not constant. The sequence is not arithmetic, even though it follows a pattern.

(e) $2\frac{1}{4} - 1\frac{1}{2} = 3 - 2\frac{1}{4} = 3\frac{3}{4} - 3 = 4\frac{1}{2} - 3\frac{3}{4} = \frac{3}{4}$, a constant. The sequence is arithmetic.

You can use a spreadsheet to check for an arithmetic sequence. First, enter the terms into a column. In the next column, starting next to the second term, enter a formula to represent $T_n - T_{n-1}$ e.g. '=A2–A1'. Fill down to copy this formula down the column, to the end of the list of terms. As this formula finds the difference between each pair of terms, it should be obvious whether or not there is a common difference.

Example 10

For the arithmetic sequence 4, 9, 14, 19, … find:

(a) the value of a **(b)** the value of d **(c)** the expression for T_n
(d) the 13th term **(e)** the value of k if $T_k = 99$.

Solution

(a) $a = 4$ **(b)** $d = 9 - 4 = 5$ **(c)** $T_n = a + (n-1)d$
$4 + 5(n-1) = 5n - 1$
$T_n = 5n - 1$
(d) $T_{13} = 4 + 12 \times 5 = 64$ **(e)** $99 = 5k - 1$
$100 = 5k$
$k = 20$

Example 11

If $T_5 = 7$ and $T_9 = -5$ are two terms of an arithmetic sequence, find the values of a and d and use them to write the first three terms of the sequence.

Solution

$T_5 = 7$: $a + 4d = 7$ [1]
$T_9 = -5$: $a + 8d = -5$ [2]
[1] − [2]: $-4d = 12$
$d = -3$
Substitute into [1]: $a - 12 = 7$
$a = 19$

The sequence begins 19, 16, 13.