ROOTS OF COMPLEX NUMBERS

Reminder: Power of a complex number using modulus-argument form (De Moivre's theorem)

Let $z = r(\cos\theta + i\sin\theta)$ then: $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$

i.e. $|z^n| = |z|^n$ and $arg(z^n) = n \times arg(z)$

This theorem was demonstrated in Lesson 2 by mathematical induction, and in Lesson 3 using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.

De Moivre's theorem can be used to find the nth root of z^n, as demonstrated in the example below:

Example 29

Find the cube roots of 1 (i.e. solve the equation $z^3 = 1$).

Solution

Let $z = r(\cos\theta + i\sin\theta)$ be a root, where r is a positive real number.

Note that $1 = 1(\cos 0 + i\sin 0)$.

Then: $r^3(\cos 3\theta + i\sin 3\theta) = 1(\cos 0 + i\sin 0)$

$\therefore$ $r^3 = 1$ (i.e. $r = 1$)

and $\cos 3\theta + i\sin 3\theta = \cos 0 + i\sin 0$

$\therefore$ $\cos 3\theta = \cos 0$ and $\sin 3\theta = \sin 0$

If you considered only $\sin 3\theta = \sin 0$, you would conclude that $3\theta = 0 + k\pi$ (where $k = 0, \pm1, \pm2, \ldots$).
But $\cos 3\theta = \cos 0$ must also be true:

$\therefore$ $3\theta = 0 + 2k\pi$ (where $k = 0, \pm1, \pm2, \ldots$)

$\therefore$ $\theta = \frac{2k\pi}{3}$

Hence: $z = 1\left(\cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}\right)$ (where $k = 0, \pm1, \pm2, \ldots$)

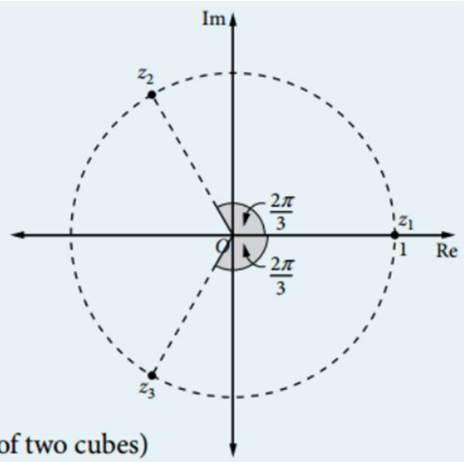
For $k = 0$: $z_1 = 1$

For $k = 1$: $z_2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

For $k = -1$: $z_3 = \cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

It might appear that there are many more values of z obtained from $k = \pm2, \pm3, \ldots$ However, if you check them you will find that they simply repeat the values of z that have already been found (because their cosines are equivalent): e.g. for $k = 2$, $z = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right) = z_3$.

There are therefore only **three** cube roots of unity, and their representation on the complex plane reveals an interesting pattern. Each has a modulus of 1, so they are all on the circumference of a unit circle.



Furthermore, the non-real roots are a pair of conjugates (as they are the roots of a polynomial equation with real coefficients), $z_2 = \overline{z_3}$. The three roots are equally spaced around the circle, each separated by an angle of $\frac{2\pi}{3}$ at the centre. They form the vertices of an equilateral triangle.

Alternatively:

The equation can be solved algebraically. $z^3 - 1 = 0$

$(z-1)(z^2 + z + 1) = 0$ (difference of two cubes)

Hence $z = 1$ or $z = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ as above.

Example 30

Find the six 6th roots of -64 (i.e. solve $z^6 = -64$).

Solution

Let $z = r(\cos\theta + i\sin\theta)$ be a root.

Then: $r^6(\cos 6\theta + i\sin 6\theta) = 64(\cos\pi + i\sin\pi)$

$\therefore$ $r^6 = 64$

$r = 2$ ($r > 0$ because it is the modulus of z)

and $6\theta = \pi + 2k\pi$ (where $k = 0, \pm1, \pm2, \ldots$)

$$\theta = \frac{\pi}{6} + \frac{2k\pi}{6}$$

Hence: $z = 2\left(\cos\left(\frac{\pi}{6} + \frac{2k\pi}{6}\right) + i\sin\left(\frac{\pi}{6} + \frac{2k\pi}{6}\right)\right)$ (where $k = 0, \pm1, \pm2, \ldots$)

For $k = 0$: $z_1 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \sqrt{3} + i$

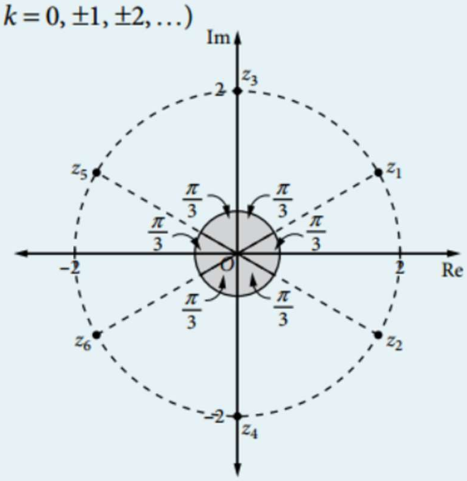
For $k = -1$: $z_2 = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) = \sqrt{3} - i$

For $k = 1$: $z_3 = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 2i$

For $k = -2$: $z_4 = 2\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right) = -2i$

For $k = 2$: $z_5 = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = -\sqrt{3} + i$

For $k = -3$: $z_6 = 2\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right) = -\sqrt{3} - i$



As each root has a modulus of 2, they are all on the circumference of a circle of radius 2. They are also equally spaced around the circumference, each separated by an angle of $\frac{2\pi}{6}$ (i.e. $\frac{\pi}{3}$) at the centre. The roots occur in conjugate pairs (as they are the roots of a polynomial equation with real coefficients). They form the vertices of a regular hexagon.

Example 31

Find the four 4th roots of $1+\sqrt{3}i$ (i.e. solve $z^4 = 1+\sqrt{3}i$). Answer in mod–arg form.

Solution

Let $z = r(\cos\theta + i\sin\theta)$ be a root.

Then: $r^4(\cos 4\theta + i\sin 4\theta) = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$\therefore$ $r^4 = 2$

$r = \sqrt[4]{2}$ ($r > 0$ because it is the modulus of z)

and $4\theta = \frac{\pi}{3} + 2k\pi$ (where $k = 0, \pm1, \pm2, \dots$)

$\theta = \frac{\pi}{12} + \frac{k\pi}{2}$

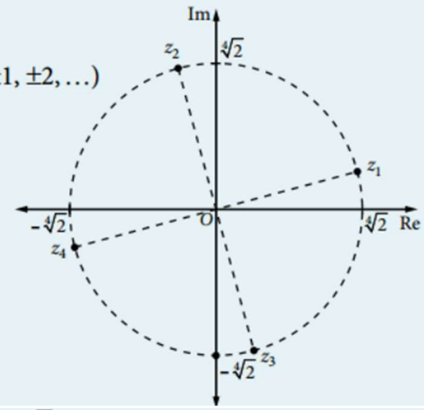
Hence: $z = \sqrt[4]{2}\left(\cos\left(\frac{\pi}{12} + \frac{k\pi}{2}\right) + i\sin\left(\frac{\pi}{12} + \frac{k\pi}{2}\right)\right)$ (where $k = 0, \pm1, \pm2, \dots$)

For $k = 0$: $z_1 = \sqrt[4]{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

For $k = -1$: $z_2 = \sqrt[4]{2}\left(\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\right)$

For $k = 1$: $z_3 = \sqrt[4]{2}\left(\cos\left(-\frac{5\pi}{12}\right) + i\sin\left(-\frac{5\pi}{12}\right)\right)$

For $k = -2$: $z_4 = \sqrt[4]{2}\left(\cos\left(-\frac{11\pi}{12}\right) + i\sin\left(-\frac{11\pi}{12}\right)\right)$



The roots are equally spaced around the circumference of a circle of radius $\sqrt[4]{2}$. Because the polynomial equation $z^4 = 1+\sqrt{3}i$ has non-real coefficients, the roots do not occur as conjugate pairs.

To summarise:

- The n[th] roots of 1 (i.e. the n roots of $z^n = 1$) are equally spaced around the circumference of the circle with centre O and radius 1, separated by an angle of $\frac{2\pi}{n}$ at the centre.
- One root is 1. If n is even, another root is (-1). The other roots occur as non-real conjugate pairs.
- The roots are $e^{\frac{2ki\pi}{n}} = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)$ where $k = 0, \pm1, \pm2, \dots$ until the n unique roots are identified.

- The n[th] roots of any complex number $Re^{i\alpha}$ are equally spaced around the circumference of the circle with centre O and radius $\sqrt[n]{R}$ separated by an angle of $\frac{2\pi}{n}$ at the centre.
- The roots are $\sqrt[n]{R}\, e^{\frac{2ki\pi}{n}} = \sqrt[n]{R}\left[\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)\right]$ where $k = 0, \pm1, \pm2, \dots$ until the n unique roots are identified.

Complex roots of unity

To investigate some properties of the n[th] roots of 1, you can work with symbolic representations rather than the actual values of the roots.

An important result that is frequently used is the factorisation:

$$w^n - 1 = (w-1)(w^{n-1} + w^{n-2} + \cdots + w^2 + w + 1)$$

For example: $w^3 - 1 = (w-1)(w^2 + w + 1)$ which you already know

Likewise $w^5 - 1 = (w-1)(w^4 + w^3 + w^2 + w + 1)$

You can prove this general factorisation by expanding the RHS.

Example 32

If w is a non-real cube root of unity, show that:

(a) $\bar{w}$ is also a root **(b)** $1 + w + \bar{w} = 0$ **(c)** $1 + w + w^2 = 0$ **(d)** $(1+w^2)^3(2+3w+3w^2) = 1$

Solution

(a) w is a root of $z^3 = 1$, so $w^3 = 1$.

Take conjugates of both sides (if two complex numbers are equal then their conjugates are equal):

$\overline{w^3} = \overline{1+0i}$

$(\bar{w})^3 = 1$ (the conjugate of a power equals the power of the conjugate)

i.e. $\bar{w}$ also satisfies $z^3 = 1$, so $\bar{w}$ is also a root.

(b) From part **(a)**, you know the three roots of $z^3 - 1 = 0$ are 1, w and $\bar{w}$.

Sum of roots $= -\dfrac{b}{a}$ $\therefore 1 + w + \bar{w} = 0$

(c) $w^3 = 1$ $\therefore w^3 - 1 = 0$

Factorise: $(w-1)(w^2 + w + 1) = 0$

$\therefore w - 1 = 0$ or $w^2 + w + 1 = 0$

But w is non-real $\therefore w^2 + w + 1 = 0$

(d) Use the results $w^3 = 1$ and $w^2 + w + 1 = 0$ to obtain $1 + w^2 = -w$ and $w + w^2 = -1$:

$$\begin{aligned}(1+w^2)^3(2+3w+3w^2) &= (-w)^3\left(2(1+w+w^2) + w + w^2\right) \\ &= -w^3(2\times 0 + (-1)) \\ &= -1 \times -1 \\ &= 1\end{aligned}$$