THE SECOND DERIVATIVE AND CONCAVITY

1 Find $f''(x)$ for each function.

(a) $f(x) = 3x^2 + 5x + 6$ **(b)** $f(x) = x^3 + 2x^2 + 4x + 2$ **(c)** $f(x) = 24 - x^2$

a) $f'(x) = 6x + 5$ $f''(x) = 6$

b) $f'(x) = 3x^2 + 4x + 4$

$f''(x) = 6x + 4$

c) $f'(x) = -2x$

$f''(x) = -2$

$\boxed{A}$ **3** Given $y = \frac{x^2 - 1}{x}$, find $\frac{d^2y}{dx^2}$. Indicate whether each statement below is a correct or incorrect step in finding $\frac{d^2y}{dx^2}$.

$\boxed{\text{(a)}}$ $y = x - \frac{1}{x}$ **(b)** $\frac{dy}{dx} = \frac{x^2 - 1}{x^2}$ $\boxed{\text{(c)}}$ $\frac{dy}{dx} = 1 + \frac{1}{x^2}$ **(d)** $\frac{d^2y}{dx^2} = \frac{-2}{x^2}$

$f(x) = \frac{u(x)}{v(x)}$ $u(x) = x^2 - 1$ $u'(x) = 2x$

$v(x) = x$ $v'(x) = 1$

$f'(x) = \frac{2x \times x - 1 \times (x^2 - 1)}{x^2} = \frac{2x^2 - x^2 + 1}{x^2} = \frac{x^2 + 1}{x^2} = 1 + \frac{1}{x^2} = 1 + x^{-2}$

$f''(x) = +(-2)x^{-3} = \frac{-2}{x^3}$

THE SECOND DERIVATIVE AND CONCAVITY

4 Find $\frac{d^2y}{dx^2}$ given: **(a)** $y=\sqrt{x}$ **(b)** $y=\sqrt{x-2}$ **(c)** $y=x\sqrt{x^2+1}$ **(d)** $y=\frac{1}{x}$

a) $y=\sqrt{x}=x^{1/2}$ $\quad f'(x)=\frac{1}{2}x^{-1/2}=\frac{1}{2\sqrt{x}}$

$f''(x)=\frac{1}{2}\left(-\frac{1}{2}\right)x^{-3/2}=-\frac{1}{4x^{3/2}}$

b) $f(x)=\sqrt{x-2}$ $\quad f'(x)=\frac{1}{2}(x-2)^{-1/2}\times 1=\frac{1}{2\sqrt{x-2}}=\frac{1}{2}(x-2)^{-1/2}$

$f''(x)=\frac{1}{2}\left(-\frac{1}{2}\right)(x-2)^{-3/2}\times 1=-\frac{1}{4(x-2)^{3/2}}$

c) $f(x)=x\sqrt{x^2+1}=u(x)\,v(x)$ $\quad u(x)=x$ $\quad u'(x)=1$

$v(x)=(x^2+1)^{1/2}$ $\quad v'(x)=\frac{1}{2}(x^2+1)^{-1/2}\times 2x$

$v'(x)=\frac{x}{(x^2+1)^{1/2}}$

$f'(x)=(x^2+1)^{1/2}+\frac{x^2}{(x^2+1)^{1/2}}$

$g(x)=\frac{x^2}{(x^2+1)^{1/2}}=\frac{u(x)}{v(x)}$ $\quad u(x)=x^2$ $\quad u'(x)=2x$

$v(x)=(x^2+1)^{1/2}$ $\quad v'(x)=\frac{x}{(x^2+1)^{1/2}}$

$g'(x)=\frac{2x(x^2+1)^{1/2}+x^3/(x^2+1)^{1/2}}{x^2+1}=\frac{2x}{(x^2+1)^{1/2}}+\frac{x^3}{(x^2+1)^{3/2}}$

So $f''(x)=\frac{x}{(x^2+1)^{1/2}}+\frac{2x}{(x^2+1)^{1/2}}+\frac{x^3}{(x^2+1)^{3/2}}=\frac{3x}{(x^2+1)^{1/2}}+\frac{x^3}{(x^2+1)^{3/2}}$

d) $f(x)=x^{-1}$ $\quad f'(x)=-1\,x^{-2}=-\frac{1}{x^2}$

$f''(x)=-(-2)\,x^{-3}=\frac{2}{x^3}$

THE SECOND DERIVATIVE AND CONCAVITY

4 Find $\dfrac{d^2y}{dx^2}$ given: (f) $y = \dfrac{x}{x+3}$ (g) $y = \dfrac{x^2+1}{\sqrt{x}}$ (i) $y = \dfrac{\sqrt{x-1}}{x+1}$

f) Quotient rule $u(x) = x$ $\quad u'(x) = 1$

$v(x) = x+3 \quad v'(x) = 1$

$$f'(x) = \frac{1 \times (x+3) - 1 \times x}{(x+3)^2} = \frac{3}{(x+3)^2} = 3(x+3)^{-2}$$

$$f''(x) = -6(x+3)^{-3} = \frac{-6}{(x+3)^3}$$

g) $$f(x) = \frac{x^2+1}{\sqrt{x}} = \frac{x^2}{\sqrt{x}} + \frac{1}{\sqrt{x}} = x^{3/2} + x^{-1/2}$$

$$f'(x) = \frac{3}{2}x^{1/2} - \frac{1}{2}x^{-3/2} \qquad f''(x) = \frac{3}{2} \times \frac{1}{2}x^{-1/2} - \frac{1}{2} \times \left(-\frac{3}{2}\right)x^{-5/2}$$

so $$f''(x) = \frac{3}{4\sqrt{x}} + \frac{3}{4x^{5/2}}$$

i) $$f(x) = \frac{\sqrt{x-1}}{x+1} = \frac{(x-1)^{1/2}}{x+1} \qquad u(x) = (x-1)^{1/2} \quad u'(x) = \frac{1}{2}(x-1)^{-1/2}$$

$$v(x) = x+1 \qquad v'(x) = 1$$

$$f'(x) = \frac{\frac{1}{2}(x-1)^{-1/2} \times (x+1) - (x-1)^{1/2}}{(x+1)^2}$$

So $$f'(x) = \frac{1}{2\sqrt{x-1}\,(x+1)} - \frac{\sqrt{x-1}}{(x+1)^2} = \frac{(x-1)^{-1/2}(x+1)^{-1}}{2} - (x-1)^{1/2}(x+1)$$

$$f''(x) = \frac{1}{2}\left[-\frac{1}{2}(x-1)^{-3/2}(x+1)^{-1} + (x-1)^{-1/2}(-1)(x+1)^{-2}\right] - \left[\frac{1}{2}(x-1)^{-1/2}(x+1)^{-2} + (x-1)^{1/2}(-2)(x+1)\right.$$

could be simplified (maybe...)

THE SECOND DERIVATIVE AND CONCAVITY

5 For what values of x is $y = 5x^2 - 1$ concave up?

$f(x) = 5x^2 - 1$ $\quad f'(x) = 10x$ $\quad f''(x) = 10$

so $f''(x) > 0$ $\quad f$ is concave up on all its domain, which is $\mathbb{R}$.

6 For what values of x is $y = 6 - 3x^2$ concave down?

$f(x) = 6 - 3x^2$ $\quad f'(x) = -6x$ $\quad f''(x) = -6$

So $f''(x) < 0$ for all x.

f is concave down on its domain, which is $\mathbb{R}$.

10 Explain why the graph of $y = \frac{1}{x^2}$ is concave up over its domain.

$f(x) = \frac{1}{x^2} = x^{-2}$ $\quad f'(x) = -2x^{-3}$ $\quad f''(x) = 6x^{-4} = \frac{6}{x^4}$

So $f''(x) > 0$ over its domain, which is $\mathbb{R} - \{0\}$.

f is concave up over its domain, which is $\mathbb{R} - \{0\}$.