INTEGRATION BY SUBSTITUTION - CHAPTER REVIEW

1 Evaluate: **(a)** $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx$ **(b)** $\int_{0}^{\frac{1}{2}} \frac{1}{1+4x^2}\,dx$ **(c)** $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\,dx$ **(d)** $\int_{0}^{\sqrt{3}} \frac{x}{\sqrt{1+x^2}}\,dx$

a) $\int_{-1/2}^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx = \left[\sin^{-1}\left(\frac{x}{1}\right)\right]_{-1/2}^{1/2} = \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(\frac{-1}{2}\right) = \frac{\pi}{6} - \left(\frac{-\pi}{6}\right) = \frac{\pi}{3}$

b) $\int_{0}^{1/2} \frac{dx}{1+4x^2} = \int_{0}^{1/2} \frac{dx}{1^2+(2x)^2} = \frac{1}{2}\int_{0}^{1/2} \frac{2\,dx}{1+(2x)^2} = \frac{1}{2} \times \left[\tan^{-1}(2x)\right]_0^{1/2} = \frac{1}{2}\tan^{-1} 1 = \frac{1}{2} \times \frac{\pi}{4} = \frac{\pi}{8}$

c) $\int_{-1/2}^{1/2} \frac{x\,dx}{\sqrt{1-x^2}} = -\frac{1}{2}\int_{-1/2}^{1/2} \frac{(-2x)\,dx}{\sqrt{1-x^2}} = -\frac{1}{2}\left[\frac{(1-x^2)^{+1/2}}{1/2}\right]_{-1/2}^{1/2}$

But actually, the function is odd, so between opposite boundaries, the definite integral is zero.

d) $\int_{0}^{\sqrt{3}} \frac{x}{\sqrt{1+x^2}}\,dx = \frac{1}{2}\int_{0}^{\sqrt{3}} \frac{2x\,dx}{\sqrt{1+x^2}} = \frac{1}{2}\left[\frac{(1+x^2)^{1/2}}{1/2}\right]_0^{\sqrt{3}}$

$\text{——} = \frac{2}{2}\left[(1+3)^{1/2} - 1^{1/2}\right]$

$\text{——} = [2-1] = 1$

2 Find: (a) $\int \frac{dx}{(1-x^2)^{\frac{3}{2}}}$ (b) $\int \frac{x^3+1}{x^2+1}dx$ (c) $\int \frac{x^3}{x^2+2x+1}dx$

a) $\int \frac{dx}{(1-x^2)^{3/2}}$ $\quad x = \sin u$ $\quad$ so $\frac{dx}{du} = \cos u$ $\quad$ so $dx = \cos u \, du$

$$\int \frac{dx}{(1-x^2)^{3/2}} = \int \frac{\cos u \, du}{(\cos^2 u)^{3/2}} = \int \frac{du}{\cos^2 u} = \int \sec^2 u \, du = \tan u + C$$

$$— = \frac{\sin u}{\cos u} + C = \frac{x}{\sqrt{1-x^2}} + C$$

b)

$$\begin{array}{r|l} & x \\ \hline x^2 & x^3 + 1 \\ & x^3 \\ \hline & 1 \end{array}$$

So $\int \frac{x^3+1}{x^2+1}dx = \int \frac{x^3}{x^2+1}dx + \int \frac{dx}{1+x^2}$

$$— = \int \frac{x(x^2+1) - x}{x^2+1} + \tan^{-1}x + C$$

$$— = \int x - \frac{x}{x^2+1} dx + \tan^{-1}x + C$$

$$— = \frac{x^2}{2} - \frac{1}{2}\ln(x^2+1) + \tan^{-1}x + C$$

c)

$$\begin{array}{r|l} & x - 2 \\ \hline x^2+2x+1 & x^3 \\ & x^3 + 2x^2 + x \\ \hline & -2x^2 - x \\ & -2x^2 - 4x - 2 \\ \hline & 3x + 2 \end{array}$$

$$I = \int \frac{x^3 \, dx}{x^2+2x+1} = \int \frac{(x^2+2x+1)(x-2) + 3x+2}{x^2+2x+1} dx$$

$$— = \int x - 2 + \frac{3x+2}{x^2+2x+1} dx$$

$$— = \frac{x^2}{2} - 2x + \int \frac{3x+2}{(x+1)^2} dx + C$$

$u = x+1$

$$I = \frac{x^2}{2} - 2x + \int \frac{3(u-1)+2}{u^2} du + C$$

$$I = \frac{x^2}{2} - 2x + 3\int \frac{du}{u} - \int \frac{du}{u^2} + C = \frac{x^2}{2} - 2x + 3\ln|x+1| + u^{-1} + C$$

So $\int \frac{x^3}{x^2+2x+1} dx = \frac{x^2}{2} - 2x + 3\ln|x+1| + \frac{1}{x+1} + C$

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3 Evaluate: (a) $\int_0^2 \frac{x}{(x^2+2)^2}dx$ (b) $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x}dx$ (c) $\int_0^1 \frac{e^x}{1+2e^x}dx$

a) $\int_0^2 \frac{x\,dx}{(x^2+2)^2} = \frac{1}{2}\int_0^2 \frac{2x}{(x^2+2)^2}dx = \frac{1}{2}\left[-\frac{1}{x^2+2}\right]_0^2 = \frac{1}{2}\left[\frac{1}{x^2+2}\right]_2^0 = \frac{1}{2}\left[\frac{1}{2}-\frac{1}{6}\right] = \frac{1}{6}$

b) $\int_0^{\pi/2} \frac{\cos x}{1+\sin x}dx = \int_0^{\pi/2} \frac{(1+\sin x)'}{1+\sin x}dx = \left[\ln|1+\sin x|\right]_0^{\pi/2}$

$= \ln\left|1+\sin\frac{\pi}{2}\right| - \ln|1+\sin 0| = \ln 2$

c) $\int_0^1 \frac{e^x}{1+2e^x}dx = \frac{1}{2}\int_0^1 \frac{2e^x}{1+2e^x}dx$

$= \frac{1}{2}\left[\ln(1+2e^x)\right]_0^1$

$= \frac{1}{2}\left[\ln(1+2e) - \ln(1+2)\right]$

$= \frac{1}{2}\ln\left(\frac{1+2e}{3}\right)$

3 Evaluate: **(d)** $\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \sin^2 x \cos^2 x \, dx$ **(e)** $\int_{\frac{3}{2}}^{4} \sqrt{2x+1}\, dx$

d) $\int_{\pi/8}^{\pi/4} \sin^2 x \cos^2 x \, dx = \frac{1}{4}\int_{\pi/8}^{\pi/4} (2\sin x \cos x)^2 \, dx = \frac{1}{4}\int_{\pi/8}^{\pi/4} (\sin 2x)^2 \, dx$

$\quad = \frac{1}{4}\int_{\pi/8}^{\pi/4} \left(\frac{1-\cos 4x}{2}\right) dx = \frac{1}{8}\int_{\pi/8}^{\pi/4} 1-\cos 4x \;\, dx$

$\quad = \frac{1}{8}\left[x - \frac{\sin 4x}{4}\right]_{\pi/8}^{\pi/4} = \frac{1}{8}\left[\left(\frac{\pi}{4} - \frac{1}{4}\times 0\right) - \left(\frac{\pi}{8} - \frac{1}{4}\times 1\right)\right]$

$\quad = \frac{1}{8}\left[\frac{\pi}{8} + \frac{1}{4}\right] = \frac{\pi + 2}{64}$

e) $\int_{3/2}^{4} \sqrt{2x+1} \, dx$

$u = 2x+1$ so $\frac{du}{dx} = 2$

$x = \frac{1}{2}(u-1)$

$\quad = \int_4^9 \sqrt{u} \; \frac{du}{2} = \frac{1}{2}\int_4^9 u^{1/2} du$

$\quad = \frac{1}{2}\left[\frac{u^{1/2+1}}{3/2}\right]_4^9 = \frac{1}{2}\times\frac{2}{3}\left[u^{3/2}\right]_4^9 = \frac{1}{3}\left[u^{3/2}\right]_4^9$

$\quad = \frac{1}{3}\left[(9^{1/2})^3 - (4^{1/2})^3\right]$

$\quad = \frac{1}{3}\left[3^3 - 2^3\right] = \frac{27-8}{3} = \frac{19}{3} = 6\frac{1}{3}$

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6 Find:

(a) $\int \frac{1-4x^2}{x} dx$ (b) $\int (\sin x + \cos x)^2 dx$ (c) $\int \sin^2 x \cos x \, dx$ (d) $\int \sin x \sec^2 x \, dx$

a) $\int \frac{1-4x^2}{x} dx = \int \frac{1}{x} - 4x \, dx = \ln|x| - 4\frac{x^2}{2} + C = \ln|x| - 2x^2 + C$

b) $\int (\sin x + \cos x)^2 dx = \int \sin^2 x + \cos^2 x + 2\sin x \cos x \; dx$

$\quad = \int 1 + \sin 2x \, dx = x - \frac{\cos 2x}{2} + C$

c) $\int \sin^2 x \cos x \; dx = \left[\frac{\sin^3 x}{3}\right] + C$

d) $\int \sin x \sec^2 x \; dx = \int \frac{\sin x}{\cos^2 x} dx = \left[\frac{\cos^{-1} x}{+1}\right] + C = \frac{\pm 1}{\cos x} + C$

$\quad = \sec x + C$

(e) $\int \frac{\sin^2 x}{\cos^2 x} dx$ (f) $\int \sin^2 x \cos^2 x \, dx$ (g) $\int \cos^2 x \, dx$ (h) $\int \cos^4 x \, dx$

e) $\int \frac{\sin^2 x}{\cos^2 x} dx = \int \tan^2 x \, dx = \int \sec^2 x - 1 \, dx = \tan x - x + C$

f) $\int \sin^2 x \cos^2 x \, dx = \frac{1}{4}\int (2 \sin x \cos x)^2 dx = \frac{1}{4}\int \sin^2 2x \, dx$

$\text{——} = \frac{1}{4}\int \frac{1 - \cos 4x}{2} dx + C = \frac{1}{8}\left[x - \frac{\sin 4x}{4}\right] + C$

$\text{——} = \frac{4x - \sin 4x}{32} + C$

g) $\int \cos^2 x \, dx = \int \frac{1 + \cos 2x}{2} dx = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right] + C$

$\text{——} = \frac{2x + \sin 2x}{4} + C$

h) $\int \cos^4 x \, dx = \int \left(\frac{1 + \cos 2x}{2}\right)^2 dx = \frac{1}{4}\int (1 + \cos 2x)^2 dx$

$\text{——} = \frac{1}{4}\int \left(1 + 2\cos 2x + \cos^2 2x\right) dx$

$\text{——} = \frac{1}{4}\left[x + 2\frac{\sin 2x}{2}\right] + C + \frac{1}{4}\int \cos^2 2x \, dx$

$\text{——} = \frac{x}{4} + \frac{\sin 2x}{4} + C + \frac{1}{4}\int \frac{1 + \cos 4x}{2} dx$

$\text{——} = \frac{x}{4} + \frac{\sin 2x}{4} + C + \frac{x}{8} + \frac{1}{8} \frac{\sin 4x}{4}$

$\text{——} = \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C$

9 Find: (a) $\int \frac{\cos 2\theta}{\sin^2 2\theta} d\theta$ (b) $\int x e^{-x^2} dx$ (c) $\int \frac{2x}{x^2+1} dx$

a) let $u = 2\theta$ $\quad \frac{du}{d\theta} = 2$ $\quad \int \frac{\cos 2\theta}{\sin^2 2\theta} d\theta = \int \frac{\cos u}{\sin^2 u} \frac{du}{2} = \frac{1}{2} \int \frac{\cos u}{\sin^2 u} du$

$= \frac{1}{2}\left[\frac{-1}{\sin u}\right] + C = -\frac{\operatorname{cosec} u}{2} + C = -\frac{\operatorname{cosec} 2\theta}{2} + C$

b) $\int x e^{-x^2} dx = -\frac{1}{2} \int (-2x) e^{-x^2} dx = -\frac{1}{2} e^{-x^2} + C$

c) $\int \frac{2x}{1+x^2} dx = \ln|1+x^2| + C$

9 Find: (d) $\int x\cos(x^2)\,dx$ (e) $\int \sec^2 x \tan^2 x\,dx$

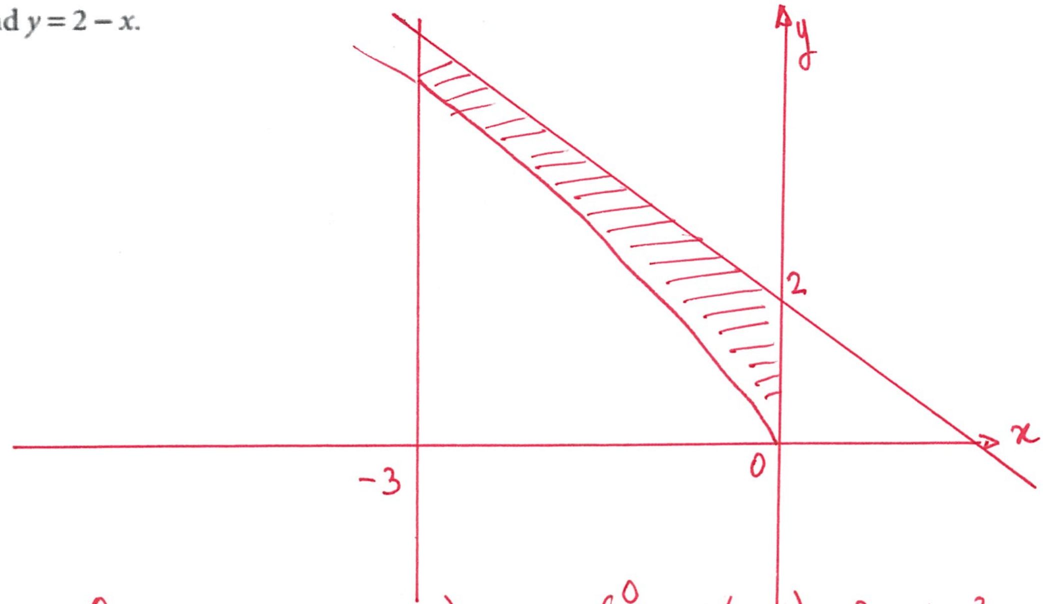
d) $\int x \cos x^2 \, dx = \frac{1}{2}\int (2x) \cos x^2 \, dx = \frac{1}{2} \sin x^2 + C$

e) $\int \sec^2 x \tan^2 x \, dx = \int (\tan x)' \tan^2 x \, dx$

$= \frac{\tan^3 x}{3} + C$

12 **(a)** Sketch the graph of the curve with equation: $y = \dfrac{x(3-x)}{x-1}$.

(b) Calculate the area of the region enclosed by the curve and the straight lines $x+3=0$, $x=0$ and $y=2-x$.



$$\text{Area} = \int_{-3}^{0} (2-x) - \left(\frac{x(3-x)}{x-1}\right) dx = \int_{-3}^{0} \frac{(2-x)(x-1) - 3x + x^2}{x-1} dx$$

$$\text{Area} = \int_{-3}^{0} \frac{3x - 3x + x^2 - x^2 - 2}{x-1} dx = -2\int_{-3}^{0} \frac{1}{x-1} dx$$

$$\text{Area} = -2\Big[\ln|x-1|\Big]_{-3}^{0}$$

$$\text{Area} = -2\Big[\ln 1 - \ln 4\Big] = 2\ln 4 \simeq 2.773 \text{ units}^2$$

17 **(a)** Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x}$.

(b) Hence find the value of $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\sin x}\,dx$.

a) $$\int_0^{\pi/2} \frac{dx}{1+\sin x} = \int_0^1 \frac{1}{1+\frac{2t}{1+t^2}} \times \frac{2\,dt}{1+t^2} = \int_0^1 \frac{2\,dt}{(1+t)^2}$$

$$= 2\left[-\frac{1}{1+t}\right]_0^1 = 2\left[\frac{1}{1+t}\right]_1^0 = 2\left[1-\frac{1}{2}\right] = 1$$

b) $$\int_0^{\pi/2} \frac{\sin x}{1+\sin x}\,dx = \int_0^{\pi/2} \frac{\sin x + 1 - 1}{1+\sin x}\,dx$$

$$= \int_0^{\pi/2} 1 - \frac{1}{1+\sin x}\,dx$$

$$= \frac{\pi}{2} - \int_0^{\pi/2} \frac{1}{1+\sin x}\,dx$$

$$= \frac{\pi}{2} - 1$$