RECURRENCE RELATIONS

1 **(a)** Differentiate $\sin x \cos^{n-1} x$ with respect to x to show that if $I_n = \int \cos^n x \, dx$, then:

$I_n = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} I_{n-2}$

$$\frac{d}{dx}\left(\sin x \cos^{n-1} x\right) = \cos x \times (\cos x)^{n-1} + \sin x \times (n-1)(\cos x)^{n-2} \times (-\sin x)$$

$$= (\cos x)^n - (n-1)\sin^2 x \, (\cos x)^{n-2}$$

$$= (\cos x)^n - (n-1)(1-\cos^2 x)(\cos x)^{n-2}$$

$$= (\cos x)^n - (n-1)(\cos x)^{n-2} + (n-1)(\cos x)^n$$

$$= n \times (\cos x)^n - (n-1)(\cos x)^{n-2}$$

$$\therefore \quad I_n = \int (\cos x)^n \, dx = \left[\int \left[\frac{d}{dx}\left(\sin x (\cos x)^{n-1}\right) + (n-1)(\cos x)^{n-2}\right] dx\right] \times \frac{1}{n}$$

$$\therefore \quad I_n = \frac{1}{n}\left[\left(\sin x \, (\cos x)^{n-1}\right) + (n-1)\int (\cos x)^{n-2} \, dx\right]$$

$$\therefore \quad I_n = \frac{1}{n} \times \sin x \, (\cos x)^{n-1} + \frac{n-1}{n} I_{n-2}$$

which is a recurrence relation.

(b) Hence evaluate: **(i)** $\int_0^{\frac{\pi}{2}} \cos^6 x\,dx$ **(iii)** $\int_0^{\frac{\pi}{2}} \cos^4 x \sin^2 x\,dx$

i) $I_6 = \int (\cos x)^6 dx = \frac{1}{6} \sin x (\cos x)^5 + \frac{5}{6} I_4$

$I_4 = \frac{1}{4} \sin x (\cos x)^3 + \frac{3}{4} I_2$

$I_2 = \int \cos^2 x\, dx = \int \frac{\cos 2x + 1}{2} dx = \frac{x}{2} + \frac{\sin 2x}{4} + C.$

$\therefore I_4 = \frac{1}{4} \sin x (\cos x)^3 + \frac{3x}{8} + \frac{3 \sin 2x}{16} + C'$

$\therefore I_6 = \frac{1}{6} \sin x (\cos x)^5 + \frac{5}{24} \sin x (\cos x)^3 + \frac{5}{16} x + \frac{5}{32} \sin 2x + C''$

$\therefore \int_0^{\pi/2} \cos^6 x\, dx = \left[\frac{1}{6} \sin x (\cos x)^5 + \frac{5}{24} \sin x (\cos x)^3 + \frac{5}{32} \sin 2x + \frac{5x}{16}\right]_0^{\pi/2}$

$\int_0^{\pi/2} (\cos x)^6 dx = 5\pi/32$

iii) $\int_0^{\pi/2} (\cos x)^4 (\sin x)^2 dx = \int_0^{\pi/2} (\cos x)^4 x - \underbrace{\int_0^{\pi/2} (\cos x)^6 dx}_{= 5\pi/32 \text{ as shown above}}$

$\int_0^{\pi/2} (\cos x)^4 dx = \left[\frac{1}{4} \sin x (\cos x)^3 + \frac{3x}{8} + \frac{3 \sin 2x}{16}\right]_0^{\pi/2}$

$\text{———} = \frac{3\pi}{16}$

$\therefore \int_0^{\pi/2} (\cos x)^4 (\sin x)^2 dx = \frac{3\pi}{16} - \frac{5\pi}{32}$

$\text{———} = \pi/32$

2 Show that $\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$ and hence find: $\int x^3 e^x\,dx$

By parts: $u(x) = x^n$ $\quad v(x) = e^x$

$u'(x) = n\,x^{n-1}$ $\quad v'(x) = e^x$

$$\int x^n e^x\,dx = x^n e^x - n\int x^{n-1} e^x\,dx$$

$$\therefore \int x^3 e^x\,dx = x^3 e^x - 3\int x^2 e^x\,dx$$

But $\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx$

But $\int x e^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C_0$

$$\therefore \int x^2 e^x\,dx = x^2 e^x - 2\left[xe^x - e^x + C_0\right]$$

$$= x^2 e^x - 2xe^x + 2e^x + C_1$$

$$\therefore \int x^3 e^x\,dx = x^3 e^x - 3\left[x^2 e^x - 2xe^x + 2e^x + C_1\right]$$

$$\therefore \int x^3 e^x\,dx = x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + C_2$$

3 **(a)** Find the derivative of $x^n \log_e x$.

(b) Hence find (correct to three decimal places) the value of:

(i) $\int_1^2 x^2 \log_e x\,dx$ **(ii)** $\int_1^2 x^3 \log_e x\,dx$

a) $\frac{d}{dx}\left(x^n \ln x\right) = n x^{n-1} \times \ln x + x^n \times \frac{1}{x} = x^{n-1}\left[n \ln x + 1\right]$

$\therefore \quad n \times x^{n-1} \ln x = \frac{d}{dx}\left(x^n \ln x\right) - x^{n-1}$

$\therefore \quad x^{n-1} \ln x = \frac{1}{n} \frac{d}{dx}\left(x^n \ln x\right) - \frac{x^{n-1}}{n}$

b) i) $\int x^2 \ln x\,dx = \int \left[\frac{1}{3} \frac{d}{dx}\left(x^3 \ln x\right) - \frac{x^2}{3}\right] dx$

$ = \frac{1}{3} x^3 \ln x - \int \frac{x^2}{3}\,dx = \frac{x^3 \ln x}{3} - \frac{x^3}{9} + C$

So $\int_1^2 x^2 \ln x\,dx = \left[\frac{x^3}{3}\left[\ln x - \frac{1}{3}\right]\right]_1^2 = \frac{8}{3}\left(\ln 2 - \frac{1}{3}\right) - \frac{1}{3}\left(-\frac{1}{3}\right) = \frac{8 \ln 2}{3} - \frac{7}{9}$

≈ 1.071

ii) $\int x^3 \ln x\,dx = \int \frac{1}{4} \frac{d}{dx}\left(x^4 \ln x\right) - \frac{x^3}{4}\,dx$

$ = \frac{1}{4} x^4 \ln x - \frac{1}{4} \int x^3\,dx$

$ = \frac{1}{4} x^4 \ln x - \frac{x^4}{16} + C$

$\therefore \int_1^2 x^3 \ln x\,dx = \left[\frac{x^4 \ln x}{4} - \frac{x^4}{16}\right]_1^2$

$ = \left(4 \ln 2 - 1\right) - \left(0 - \frac{1}{16}\right)$

$ = 4 \ln 2 - \frac{15}{16} \approx 1.835$

4 **(a)** Given that $I_n = \int_0^{\frac{\pi}{2}} \cos^n x\, dx$, prove that $I_n = \left(\frac{n-1}{n}\right) I_{n-2}$ where $n \geq 2$ is an integer.

(b) Hence evaluate: $\int_0^{\frac{\pi}{2}} \cos^5 x\, dx$

a) $I_n = \int_0^{\pi/2} (\cos x)^n dx = \int_0^{\pi/2} \cos x \times (\cos x)^{n-1} dx$ integration by parts

$u(x) = (\cos x)^{n-1}$ $v(x) = \sin x$

$u'(x) = (n-1)(\cos x)^{n-2} \times (-\sin x)$ $v'(x) = \cos x$

$\therefore I_n = \left[\sin x \times (\cos x)^{n-1}\right]_0^{\pi/2} - \int_0^{\pi/2} (n-1)(\cos x)^{n-2}(-\sin x) \sin x\, dx$

$\therefore I_n = (n-1) \int_0^{\pi/2} \sin^2 x\, (\cos x)^{n-2} dx$

$\therefore I_n = (n-1) \int_0^{\pi/2} (1-\cos^2 x)(\cos x)^{n-2} dx$

$I_n = (n-1)\left[\int_0^{\pi/2} (\cos x)^{n-2} dx - I_n\right]$

so $I_n = (n-1) I_{n-2} - (n-1) I_n$

$I_n(1+(n-1)) = (n-1) I_{n-2}$ $\therefore I_n = \left(\frac{n-1}{n}\right) I_{n-2}$

b) $\int_0^{\pi/2} (\cos x)^5 dx = I_5 = \frac{4}{5} I_3 = \frac{4}{5} \int_0^{\pi/2} (\cos x)^3 dx$

$\int_0^{\pi/2} (\cos x)^3 dx = \int_0^{\pi/2} \cos x (1-\sin^2 x) dx = \int_0^{\pi/2} \cos x\, dx - \int_0^{\pi/2} \cos x \sin^2 x\, dx$

$= \left[\sin x\right]_0^{\pi/2} - \left[\frac{\sin^3 x}{3}\right]_0^{\pi/2} = 1 - \frac{1}{3} = \frac{2}{3}$

$\therefore \int_0^{\pi/2} (\cos x)^5 dx = \frac{4}{5} \times \frac{2}{3} = \frac{8}{15}$

5 **(a)** Show that: $\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$

(b) Hence evaluate: $\int_0^{\frac{\pi}{6}} \sin^4 x\,dx$

a) $\int (\sin x)^n dx = \int \sin x \times (\sin x)^{n-1} dx$ we integrate by parts.

$u(x) = (\sin x)^{n-1}$ $\quad$ $v(x) = -\cos x$

$u'(x) = (n-1)(\sin x)^{n-2} \times \cos x$ $\quad$ $v'(x) = \sin x$

$$\int (\sin x)^n dx = (\sin x)^{n-1}(-\cos x) - \int (n-1)(\sin x)^{n-2}\cos x \times (-\cos x)\,dx$$

$$\text{——} = -\cos x\,(\sin x)^{n-1} + (n-1)\int \cos^2 x\,(\sin x)^{n-2}\,dx$$

$$\text{——} = -\cos x\,(\sin x)^{n-1} + (n-1)\int (1-\sin^2 x)(\sin x)^{n-2}\,dx$$

$$\text{——} = -\cos x\,(\sin x)^{n-1} + (n-1)\left[\int (\sin x)^{n-2}dx - \int (\sin x)^n dx\right]$$

$$\therefore \left[1+(n-1)\right]\int (\sin x)^n dx = -\cos x\,(\sin x)^{n-1} + (n-1)\int (\sin x)^{n-2}dx$$

$$\therefore \int (\sin x)^n dx = -\frac{(\sin x)^{n-1}\cos x}{n} + \frac{n-1}{n}\int (\sin x)^{n-2}dx$$

b) $$\int (\sin x)^4 dx = -\frac{(\sin x)^3\cos x}{4} + \frac{3}{4}\int \sin^2 x\,dx$$

$$\text{——} = -\frac{(\sin x)^3\cos x}{4} + \frac{3}{4}\int \frac{1-\cos 2x}{2}\,dx$$

$$\text{——} = -\frac{(\sin x)^3\cos x}{4} + \frac{3}{8}x - \frac{3}{8}\,\frac{\sin 2x}{2} + C$$

$$\therefore \int_0^{\pi/6} (\sin x)^4 dx = \left[-\frac{(\sin x)^3\cos x}{4} + \frac{3}{8}x - \frac{3\sin 2x}{16}\right]_0^{\pi/6}$$

$$\text{——} = -\frac{(1/8)\times\frac{\sqrt{3}}{2}}{4} + \frac{\pi}{16} - \frac{3\times(\sqrt{3}/2)}{16} = \frac{\pi}{16} - \sqrt{3}\left(\frac{1}{64}+\frac{3}{32}\right) = \frac{\pi}{16} - \frac{7\sqrt{3}}{64}$$

$\cos 2\theta = 1 - 2\sin^2\theta$

6 **(a)** If $I_n = \int \sec^n x\,dx$ show that: $I_n = \frac{1}{n-1}(\sec^{n-2} x \tan x) + \frac{n-2}{n-1} I_{n-2}$

(b) Hence evaluate: $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^4 x\,dx$

a) $\int (\sec x)^n dx = \int \sec^2 x \times (\sec x)^{n-2} dx$ we integrate by parts.

$u(x) = (\sec x)^{n-2}$ $\quad v(x) = \tan x$

$u'(x) = (n-2)(\sec x)^{n-3} \times (\sec x)'$ $\quad v'(x) = \sec^2 x$

$(\sec x)' = \left(\frac{1}{\cos x}\right)' = ((\cos x)^{-1})' = (-1)(\cos x)^{-2} \times (-\sin x) = \frac{\sin x}{\cos^2 x} = \tan x \sec x$

so $u'(x) = (n-2)(\sec x)^{n-3} \times \tan x \sec x = (n-2) \tan x (\sec x)^{n-2}$

$\therefore \int (\sec x)^n dx = \tan x \times (\sec x)^{n-2} - \int (n-2) \tan^2 x (\sec x)^{n-2} dx$

But $1 + \tan^2 x = \sec^2 x$, $\therefore \tan^2 x = \sec^2 x - 1$

$\therefore \int (\sec x)^n dx = \tan x \times (\sec x)^{n-2} - (n-2) \int (\sec^2 x - 1)(\sec x)^{n-2} dx$

$\therefore \int (\sec x)^n dx = \tan x \times (\sec x)^{n-2} - (n-2) \int (\sec x)^n dx + (n-2) \int (\sec x)^{n-2} dx$

$\therefore I_n [1 + (n-2)] = \tan x \times (\sec x)^{n-2} + (n-2) I_{n-2}$

$\therefore I_n = \frac{1}{n-1} \tan x (\sec x)^{n-2} + \left(\frac{n-2}{n-1}\right) I_{n-2}$

b) $\int \sec^4 x\,dx = \frac{1}{3} \tan x \times \sec^2 x + \frac{2}{3} \int \sec^2 x\,dx$

$\_\_\_ = \frac{1}{3} \tan x \sec^2 x + \frac{2}{3} \tan x + C$

$\therefore \int_{\pi/6}^{\pi/3} \sec^4 x\,dx = \left[\frac{1}{3} \tan x \sec^2 x + \frac{2}{3} \tan x\right]_{\pi/6}^{\pi/3}$

$\_\_\_ = \left(\frac{1}{3}\sqrt{3} \times 4 + \frac{2}{3}\sqrt{3}\right) - \left(\frac{1}{3} \times \frac{1}{\sqrt{3}} \times \frac{4}{3} + \frac{2}{3} \times \frac{1}{\sqrt{3}}\right) = 2\sqrt{3} - \frac{1}{\sqrt{3}}\left(\frac{10}{9}\right)$

$\_\_\_ = 2\sqrt{3} - \frac{10\sqrt{3}}{27} = \frac{44\sqrt{3}}{27}$

8 **(a)** Given that $I_n = \int_0^1 x^{2n-1} e^{x^2}\,dx$ for each integer $n \geq 1$, show that: $I_n = \frac{e}{2} - (n-1)I_{n-1}$

(b) Hence, or otherwise, calculate I_2.

a) $I_n = \int_0^1 x^{2n-1} e^{x^2}\,dx$ we integrate by parts.

$u(x) = e^{x^2}$ $\qquad v(x) = \frac{x^{2n}}{2n}$

$u'(x) = e^{x^2} \times 2x$ $\qquad v'(x) = x^{2n-1}$

$$\text{So } I_n = \left[\frac{e^{x^2} \times x^{2n}}{2n}\right]_0^1 - \int_0^1 \frac{e^{x^2} \times 2x \times x^{2n}}{2n}\,dx$$

$$I_n = \frac{e}{2n} - \frac{1}{n}\int_0^1 e^{x^2} x^{2n+1}\,dx.$$

$$I_n = \frac{e}{2n} - \frac{1}{n} I_{n+1}$$

$$\therefore \frac{1}{n} I_{n+1} = \frac{e}{2n} - I_n \Leftrightarrow I_{n+1} = \frac{e}{2} - nI_n$$

$$\text{or } I_n = \frac{e}{2} - (n-1)I_{n-1}$$

b) $I_2 = \frac{e}{2} - (2-1)I_1 = \frac{e}{2} - I_1$

But $I_1 = \int_0^1 x e^{x^2}\,dx = \frac{1}{2}\int_0^1 2x e^{x^2}\,dx = \frac{1}{2}\left[e^{x^2}\right]_0^1 = \frac{1}{2}\left[e-1\right]$

$$\therefore I_2 = \frac{e}{2} - \left(\frac{1}{2}(e-1)\right) = \frac{1}{2}$$

10 **(a)** Show that: $\int e^{ax}\sin x\,dx = \frac{1}{a^2+1}e^{ax}(a\sin x - \cos x)$

(b) Hence find: **(i)** $\int e^{x}\sin x\,dx$ **(ii)** $\int e^{3x}\sin x\,dx$

a) $u(x) = \sin x$ $\quad v(x) = \frac{e^{ax}}{a}$ by parts.

$u'(x) = \cos x$ $\quad v'(x) = e^{ax}$

$$\int e^{ax}\sin x\,dx = \left(\frac{\sin x\, e^{ax}}{a}\right) - \int \frac{\cos x\, e^{ax}}{a}\,dx.$$

we integrate by parts again.

$u(x) = \cos x$ $\quad v(x) = \frac{e^{ax}}{a}$

$u'(x) = -\sin x$ $\quad v'(x) = e^{ax}$

$$\int e^{ax}\sin x\,dx = \left(\frac{\sin x\, e^{ax}}{a}\right) - \frac{1}{a}\left[\frac{\cos x\, e^{ax}}{a} + \int \frac{e^{ax}\sin x}{a}\,dx\right]$$

$$\left[\int e^{ax}\sin x\,dx\right] \times \left(1 + \frac{1}{a^2}\right) = \frac{1}{a^2}\left[a\sin x\, e^{ax} - \cos x\, e^{ax}\right]$$

$$\frac{a^2+1}{a^2}\int e^{ax}\sin x\,dx = \frac{1}{a^2}\left[a\sin x\, e^{ax} - \cos x\, e^{ax}\right] + C$$

$$\therefore \int e^{ax}\sin x\,dx = \frac{1}{a^2+1}\left(a\sin x\, e^{ax} - \cos x\, e^{ax}\right) + C$$

b) i) $\int e^{x}\sin x\,dx = \frac{1}{2}\left(\sin x\, e^{x} - \cos x\, e^{x}\right) = \frac{e^x}{2}\left(\sin x - \cos x\right) + C$

ii) $\int e^{3x}\sin x\,dx = \frac{1}{10}\left(3\sin x\, e^{3x} - \cos x\, e^{3x}\right) + C$

$$\int e^{3x}\sin x\,dx = \frac{e^{3x}}{10}\left(3\sin x - \cos x\right) + C.$$

11 If $I_n = \int_0^x \frac{t^n}{1+t}\,dt$ show that: $I_n + I_{n-1} = \frac{x^n}{n}$

$$I_n + I_{n-1} = \int_0^x \frac{t^n}{1+t}\,dt + \int_0^x \frac{t^{n-1}}{1+t}\,dt$$

$$\text{———} = \int_0^x \frac{t^n + t^{n-1}}{1+t}\,dt = \int_0^x t^{n-1}\frac{(t+1)}{(1+t)}\,dt$$

$$\therefore\ I_n + I_{n-1} = \int_0^x t^{n-1}\,dt$$

$$\text{———} = \left[\frac{t^n}{n}\right]_0^x$$

$$\therefore\ I_n + I_{n-1} = \frac{x^n}{n}$$

This last one was the easiest! ☺