OVERVIEW OF TRIGONOMETRIC EQUATIONS

1 Solve for $0 \le x \le 2\pi$.

(a) $\sin x = 1$ (b) $\cos x = 0$ (c) $\tan x = -1$ (d) $\sqrt{3}\operatorname{cosec} x = 2$ (e) $\sec x = -2$

(f) $\cot x = \sqrt{3}$ (g) $2\sin\left(x - \frac{\pi}{6}\right) + 1 = 0$ (h) $\cos\frac{x}{2} = 1$ (i) $2\sin^2 x = 1$ (j) $\sin x = 0.3894$

a) $\sin x = 1 = \sin\frac{\pi}{2}$ so $x = \pi/2$

b) $\cos x = 0$ so $x = \pi/2$ or $x = \frac{3\pi}{2}$

c) $\tan x = -1 = \tan\frac{3\pi}{4}$ so $x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$

d) $\sqrt{3}\operatorname{cosec} x = 2 \iff \frac{\sqrt{3}}{\sin x} = 2 \iff \sin x = \frac{\sqrt{3}}{2}$ so $x = \pi/3$ or $x = 2\pi/3$

e) $\sec x = -2 \iff \frac{1}{\cos x} = -2 \iff \cos x = -\frac{1}{2}$ $x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$

f) $\cot x = \sqrt{3} \iff \frac{\cos x}{\sin x} = \sqrt{3} \iff \frac{\sin x}{\cos x} = \frac{\sqrt{3}}{3}$ $\tan x = \frac{1}{\sqrt{3}}$ $x = \pi/6$ or $x = \frac{7\pi}{6}$

g) $2\sin\left(x - \frac{\pi}{6}\right) = -1 \iff \sin\left(x - \frac{\pi}{6}\right) = -\frac{1}{2} = \sin\frac{7\pi}{6}$

so $x - \frac{\pi}{6} = (-1)^n\frac{7\pi}{6} + n\pi$ $n = 0$ gives $x = \frac{7\pi}{6} + \frac{\pi}{6} = \frac{4\pi}{3}$ $x = \frac{4\pi}{3}$

$n = 1$ gives $x = -\frac{7\pi}{6} + \pi + \frac{\pi}{6} = 0$ $x = 0$

h) $\cos\frac{x}{2} = 1 = \cos 0$ so $\frac{x}{2} = 0$ so $x = 0$

i) $2\sin^2 x = 1 \iff \sin^2 x = \frac{1}{2} \iff \sin x = \frac{\sqrt{2}}{2}$ or $\sin x = -\frac{\sqrt{2}}{2}$

$x = \frac{\pi}{4}$ or $x = \frac{3\pi}{4}$ or $x = \frac{5\pi}{4}$ or $x = \frac{7\pi}{4}$

j) $\sin x = 0.3894$ $x = (-1)^n \times \sin^{-1} 0.3894 + n\pi$

$= \sin\theta$ $n = 0$ gives $x = \sin^{-1} 0.3894$

$\theta = \sin^{-1} 0.3894$ $n = 1$ gives $x = \pi - \sin^{-1}(0.3894)$

OVERVIEW OF TRIGONOMETRIC EQUATIONS

3 Solve for $-\pi \leq x \leq \pi$.

(a) $\cos^2 x - 2\cos x + 1 = 0$ **(b)** $\sin^2 x = \sin x$ **(c)** $\cos 2x = \sin x$

a) let do a change of variable $X = \cos x$

so the equation becomes $X^2 - 2X + 1 = 0 \iff (X-1)^2 = 0$

so $X = 1$ so $\cos x = 1$ $x = 0$

b) $\sin^2 x = \sin x \iff \sin^2 x - \sin x = 0 \iff \sin x(\sin x - 1) = 0$

either $\sin x = 0 \implies x = 0$ or $x = +\pi$ or $x = -\pi$

or $\sin x = 1 \implies x = \frac{\pi}{2}$

So 4 solutions : $x = -\pi$ $x = 0$ $x = \frac{\pi}{2}$ $x = \pi$

c) $\cos 2x = \sin x \iff 1 - 2\sin^2 x = \sin x$

$\iff -2\sin^2 x - \sin x + 1 = 0$

$\iff 2\sin^2 x + \sin x - 1 = 0$ $X = \sin x$

$\iff 2X^2 + X - 1 = 0$ $\Delta = 1 + 4 \times 2 = 9 = 3^2$

$X_1 = \frac{-1+3}{4} = \frac{1}{2}$ or $X_2 = \frac{-1-3}{4} = -1$

So either $\sin x = \frac{1}{2}$, i.e. $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$

OR $\sin x = -1$, i.e $x = \frac{-\pi}{2}$

3 solutions $x = -\frac{\pi}{2}$, $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$

OVERVIEW OF TRIGONOMETRIC EQUATIONS

3 Solve for $-\pi \le x \le \pi$.

(d) $\sin^2 x = 1 - \cos x$ **(e)** $\cos 2x = 2 + \cos x$ **(f)** $\tan 2x = \cot x$

d) $\Leftrightarrow 1 - \cos^2 x = 1 - \cos x \Leftrightarrow -\cos^2 x + \cos x = 0$

$\Leftrightarrow \cos^2 x - \cos x = 0$

$\Leftrightarrow \cos x(\cos x - 1) = 0$

either $\cos x = 0$ i.e $x = -\frac{\pi}{2}$ or $x = \frac{\pi}{2}$

OR $\cos x = 1$ i.e $x = 0$

So 3 solutions $x = -\frac{\pi}{2}$, $x = 0$ or $x = \frac{\pi}{2}$

e) $\cos 2x = 2 + \cos x \Leftrightarrow 2\cos^2 x - 1 = 2 + \cos x \Leftrightarrow 2\cos^2 x - \cos x - 3 = 0$

let $X = \cos x$ so the equation becomes $2X^2 - X - 3 = 0$

$\Delta = 1 - 4 \times (-3) \times 2 = 25 = 5^2$ $X = \frac{1-5}{4} = -1$ or $X = \frac{1+5}{4} = \frac{3}{2}$

$\cos x = \frac{3}{2}$ is impossible as $-1 \le \cos x \le 1$

$\cos x = -1$ means $x = -\pi$ or $x = \pi$

f) $\tan 2x = \cot x \Leftrightarrow \frac{2\tan x}{1 - \tan^2 x} = \frac{1}{\tan x} \Leftrightarrow 2\tan^2 x = 1 - \tan^2 x$

$\Leftrightarrow 3\tan^2 x = 1$

$\Rightarrow \tan^2 x = \frac{1}{3}$ so $\tan x = \frac{1}{\sqrt{3}}$ OR $\tan x = -\frac{1}{\sqrt{3}}$

if $\tan x = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ means $x = \pi/6$ OR $x = -\frac{5\pi}{6}$

for $\tan x = -\frac{1}{\sqrt{3}}$ means $x = \frac{5\pi}{6}$ OR $x = \frac{-\pi}{6}$

So 4 solutions: $x = -\frac{5\pi}{6}, \frac{-\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$

OVERVIEW OF TRIGONOMETRIC EQUATIONS

5 Solve for $0 \le \theta \le 2\pi$.

(a) $\sqrt{2}\sin 2\theta + 1 = 0$ (b) $\tan\left(\theta - \frac{\pi}{3}\right) = -\sqrt{3}$ (c) $\cos 2\theta \cos\frac{\pi}{6} - \sin 2\theta \sin\frac{\pi}{6} = 0.5$

a) $\Leftrightarrow \sin 2\theta = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} = \sin\frac{5\pi}{4}$

General solution is $2\theta = (-1)^n \times \frac{5\pi}{4} + n\pi \Rightarrow \theta = (-1)^n \frac{5\pi}{8} + n\frac{\pi}{2}$

$n = 0 \Rightarrow \theta = 5\pi/8$ $n = 1$ means $\theta = \frac{\pi}{2} - \frac{5\pi}{8} = -\frac{\pi}{8}$ outside

$n = 2$ means $\theta = \pi + \frac{5\pi}{8} = \frac{13\pi}{8}$ $n = 3$ gives $\theta = \frac{3\pi}{2} - \frac{5\pi}{8} = \frac{7\pi}{8}$

$n = 4$ outside $n = 5$ gives $\theta = \frac{5\pi}{2} - \frac{5\pi}{8} = \frac{15\pi}{8}$

Other values are outside of the range. So $\frac{5\pi}{8}, \frac{7\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$

b) $\tan\left(\theta - \frac{\pi}{3}\right) = \tan\frac{2\pi}{3}$ General solution is $\theta - \frac{\pi}{3} = \frac{2\pi}{3} + n\pi$

or $\theta = \pi + n\pi$ for $n = -1$ $\theta = 0$

For $n = 0$ $\theta = \pi$ for $n = 1$ $\theta = 2\pi$

So $0, \pi, 2\pi$

c) $\Leftrightarrow \cos\left(2\theta + \frac{\pi}{6}\right) = \frac{1}{2} = \cos\frac{\pi}{3}$ General solution is $2\theta + \frac{\pi}{6} = \pm\frac{\pi}{3} + 2n\pi$

or $2\theta = \pm\frac{\pi}{3} - \frac{\pi}{6} + 2n\pi$ or $\theta = \pm\frac{\pi}{6} - \frac{\pi}{12} + n\pi$

For $n = 0$ $\theta = \frac{\pi}{6} - \frac{\pi}{12} = \frac{\pi}{12}$ and $\theta = -\frac{\pi}{6} - \frac{\pi}{12} = \frac{-3\pi}{12} = -\frac{\pi}{4}$ (outside)

For $n = 1$ $\theta = \frac{\pi}{6} - \frac{\pi}{12} + \pi = \frac{13\pi}{12}$ and $\theta = -\frac{\pi}{4} + \pi = \frac{9\pi}{12} = \frac{3\pi}{4}$

For $n = 2$ $\theta = \frac{\pi}{12} + 2\pi$ outside and $\theta = -\frac{\pi}{4} + 2\pi = \frac{7\pi}{4}$

For $n = -1$ $\theta = \frac{\pi}{12} - \pi$ outside and $\theta = -\frac{\pi}{4} - \pi$ outside.

So $\frac{\pi}{12}, \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{13\pi}{12}$

6 Solve for $-\pi \leq \theta \leq \pi$.

(a) $\cos 3\theta = \cos\theta$ (b) $2\cos 2\theta = 4\cos\theta - 3$ (c) $3\tan 2\theta = 2\tan\theta$

a) General solution is $3\theta = \pm\theta + 2n\pi$

① if $3\theta = \theta + 2n\pi$ then $2\theta = 2n\pi$ so $\theta = n\pi$

so either $\theta = -\pi$, $\theta = 0$ or $\theta = \pi$

② if $3\theta = -\theta + 2n\pi$ then $4\theta = 2n\pi$ so $\theta = n\frac{\pi}{2}$

$\theta = -\pi$ or $\theta = -\frac{\pi}{2}$ or $\theta = 0$ or $\theta = \frac{\pi}{2}$ or $\theta = \pi$

in Summary : $-\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$

b) $\Leftrightarrow 2[2\cos^2\theta - 1] = 4\cos\theta - 3 \Leftrightarrow 4\cos^2\theta - 4\cos\theta + 1 = 0$

$\Leftrightarrow \cos^2\theta - \cos\theta + \frac{1}{4} = 0 \Leftrightarrow \left(\cos\theta - \frac{1}{2}\right)^2 = 0$

so $\cos\theta = \frac{1}{2} = \cos\frac{\pi}{3}$ $\theta = \pm\frac{\pi}{3} + 2n\pi$

$\theta = -\frac{\pi}{3}$; $\theta = \frac{\pi}{3}$; others solutions are outside of $[-\pi, \pi]$

c) $\Leftrightarrow \frac{6\tan\theta}{1 - \tan^2\theta} = 2\tan\theta \Leftrightarrow 6\tan\theta = 2\tan\theta - 2\tan^3\theta$

$\Leftrightarrow 2\tan^3\theta + 4\tan\theta = 0$

$\Leftrightarrow \tan^3\theta + 2\tan\theta = 0 \Leftrightarrow \tan\theta\,[\underbrace{\tan^2\theta + 2}_{\text{cannot be zero as always positive.}}] = 0$

So $\tan\theta = 0 = \tan 0$

General solution is $\theta = 0 + n\pi$

So $-\pi, 0, \pi$

OVERVIEW OF TRIGONOMETRIC EQUATIONS

6 Solve for $-\pi \le \theta \le \pi$.

(d) $\tan\left(2\theta - \frac{\pi}{4}\right) + 1 = 0$ (e) $2\cos\left(2\theta - \frac{\pi}{3}\right) = \sqrt{3}$ (f) $2\sin^2\theta + \cos\theta = 1$

d) $\Leftrightarrow \tan\left(2\theta - \frac{\pi}{4}\right) = -1 = \tan\left(\frac{3\pi}{4}\right)$ General solution is $2\theta - \frac{\pi}{4} = \frac{3\pi}{4} + n\pi$

or $2\theta = \pi + n\pi$ or $\theta = \frac{\pi}{2} + n\frac{\pi}{2}$

$n=-3$ gives $\theta = -\pi$; $n=-2$ gives $\theta = -\frac{\pi}{2}$; $n=-1$ gives $\theta = 0$

$n=0$ gives $\pi/2$

$n=1$ gives $\theta = \pi$; others solutions are outside the interval $[-\pi, \pi]$

So $-\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$

e) $\Leftrightarrow \cos\left(2\theta - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}$ General solution is $2\theta - \frac{\pi}{3} = \pm\frac{\pi}{6} + 2n\pi$

or $2\theta = \frac{\pi}{3} \pm \frac{\pi}{6} + 2n\pi$ or $\theta = \frac{\pi}{6} \pm \frac{\pi}{12} + n\pi$

① if $\theta = \frac{\pi}{6} + \frac{\pi}{12} + n\pi = \frac{3\pi}{12} + n\pi = \frac{\pi}{4} + n\pi$

$n=-1$ gives $\theta = -\frac{3\pi}{4}$; $n=0$ gives $\theta = \frac{\pi}{4}$;

② if $\theta = \frac{\pi}{6} - \frac{\pi}{12} + n\pi = \frac{\pi}{12} + n\pi$

$n=-1$ gives $\theta = -\frac{11}{12}\pi$; $n=0$ gives $\theta = \frac{\pi}{12}$;

So $-\frac{11\pi}{12}$; $-\frac{3\pi}{4}$; $\frac{\pi}{12}$, $\frac{\pi}{4}$

f) $\Leftrightarrow 2(1-\cos^2\theta) + \cos\theta - 1 = 0 \Leftrightarrow -2\cos^2\theta + \cos\theta + 1 = 0$

or $-2X^2 + X + 1 = 0$ $\Delta = 1 - 4\times(-2) = 9 = 3^2$

$X = \frac{-1+3}{(-4)} = -\frac{1}{2}$ or $X = \frac{-1-3}{-4} = 1$

① $\cos X = -\frac{1}{2} = \cos\left(\frac{2\pi}{3}\right)$ so $x = \pm\frac{2\pi}{3} + 2n\pi$

if $x = \frac{2\pi}{3} + 2n\pi$ $n=0$ gives $x = \frac{2\pi}{3}$ | if $x = -\frac{2\pi}{3} + 2n\pi$ $n=0$ gives $x = -\frac{2\pi}{3}$

② $\cos x = 1 = \cos 0$ so $x = 2n\pi$

$n=0$ gives $x=0$

So $-\frac{2\pi}{3}$; 0; $\frac{2\pi}{3}$

OVERVIEW OF TRIGONOMETRIC EQUATIONS

8 Solve for $0 \le \theta \le 2\pi$.

(a) $\tan^3\theta - \tan\theta = 0$ (b) $\tan\theta = \sin\theta$ (c) $\sec 2\theta = \operatorname{cosec} 2\theta$

a) $\Leftrightarrow \tan\theta(\tan^2\theta - 1) = 0$ either $\tan\theta = 0$ or $\tan^2\theta = 1$

if $\tan\theta = 0$ then $\theta = 0 + n\pi$ so possible values are $0, \pi, 2\pi$

if $\tan^2\theta = 1$ then $\theta = \frac{\pi}{4} + n\pi$ possible values are $\frac{\pi}{4}$; $\frac{5\pi}{4}$

or $\tan\theta = -1$ then $\theta = -\frac{\pi}{4} + n\pi$ $\theta = \frac{3\pi}{4}$; $\theta = \frac{7\pi}{4}$

In summary : $0; \frac{\pi}{4}; \frac{3\pi}{4}; \frac{5\pi}{4}; \pi; \frac{7\pi}{4}; 2\pi$

b) $\Leftrightarrow \sin\theta\left[\frac{1}{\cos\theta} - 1\right] = 0$ either $\sin\theta = 0$ or $\cos\theta = 1$

if $\sin\theta = 0$ then $\theta = 0$ or π or 2π

for $\cos\theta = 1$ then $\theta = 0$ or $\theta = 2\pi$

In summary $0, \pi, 2\pi$

c) $\Leftrightarrow \frac{1}{\cos 2\theta} = \frac{1}{\sin 2\theta} \Leftrightarrow \sin 2\theta = \cos 2\theta \Leftrightarrow \tan 2\theta = 1$

So $2\theta = \frac{\pi}{4} + n\pi$ or $\theta = \frac{\pi}{8} + n\frac{\pi}{2}$

$n = 0$ gives $\theta = \pi/8$ $n = 1$ gives $\theta = \frac{5\pi}{8}$

$n = 2$ gives $\theta = \frac{9\pi}{8}$ $n = 3$ gives $\theta = \frac{13\pi}{8}$

other solutions are outside of the range.

So $\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$

OVERVIEW OF TRIGONOMETRIC EQUATIONS

8 Solve for $0 \le \theta \le 2\pi$.

(d) $\sin 2\theta = \tan\theta$ (e) $\sin 3\theta = \sin 2\theta$

d) $2\sin\theta\cos\theta - \dfrac{\sin\theta}{\cos\theta} = 0 \iff \sin\theta\left[2\cos\theta - \dfrac{1}{\cos\theta}\right] = 0$

So either $\sin\theta = 0$, i.e. $\theta = 0$, $\theta = \pi$ or $\theta = 2\pi$

or $2\cos\theta = \dfrac{1}{\cos\theta}$, i.e. $\cos^2\theta = \dfrac{1}{2}$ so $\cos\theta = \pm\dfrac{\sqrt{2}}{2}$

* if $\cos\theta = \dfrac{\sqrt{2}}{2} = \cos\dfrac{\pi}{4}$ general solution is $\theta = \pm\dfrac{\pi}{4} + 2n\pi$

$\theta = \dfrac{\pi}{4} + 2n\pi$ $n=0$ gives $\theta = \dfrac{\pi}{4}$

$\theta = -\dfrac{\pi}{4} + 2n\pi$ $n=1$ gives $\theta = \dfrac{7\pi}{4}$

* if $\cos\theta = -\dfrac{\sqrt{2}}{2} = \cos\dfrac{3\pi}{4}$ general solution is $\theta = \pm\dfrac{3\pi}{4} + 2n\pi$

$\theta = \dfrac{3\pi}{4} + 2n\pi$ $n=0$ gives $\theta = \dfrac{3\pi}{4}$

$\theta = -\dfrac{3\pi}{4} + 2n\pi$ $n=1$ gives $\theta = \dfrac{5\pi}{4}$

So: $0, \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \pi, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}, 2\pi$

e) $3\theta = (-1)^n \times 2\theta + n\pi \iff \theta\left[3 - 2(-1)^n\right] = n\pi$

$\theta = \dfrac{n\pi}{(3 - 2(-1)^n)}$

$n = -1$ gives $\theta = \dfrac{-\pi}{3 - 2\times(-1)} = -\dfrac{\pi}{5}$ outside

$n=-2$ gives $\theta = \dfrac{-2\pi}{1}$ outside

$n = 0$ gives $\theta = 0$

$n = 1$ gives $\theta = \dfrac{\pi}{5}$

$n=3$ gives $\theta = \dfrac{3\pi}{5}$

$n = 2$ gives $\theta = 2\pi$

$n = 5$ gives $\theta = \pi$ $n = 7$ gives $x = \dfrac{7\pi}{5}$

$n = 9$ gives $\theta = \dfrac{9\pi}{5}$

So $0, \dfrac{\pi}{5}, \dfrac{3\pi}{5}, \pi, \dfrac{7\pi}{5}, \dfrac{9\pi}{5}, 2\pi$