THE BINOMIAL DISTRIBUTION - CHAPTER REVIEW (CAMBRIDGE)

1 A marksman finds that on average he hits the target five times out of six. Assuming that successive shots are independent events, find the probability that in four shots:

a he has exactly three hits, **b** he has exactly two misses.

a) $P(X=3) = {}^4C_3\left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right) = \frac{125}{324} \approx 0.385$

b) $P(X=2) = {}^4C_2\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)^2 = \frac{150}{1296} = \frac{25}{216} \approx 11.6\%$

2 Five out of six people surveyed think that Tasmania is the most beautiful state in Australia. What is the probability that in a group of 15 randomly selected people, at least 13 of them think that Tasmania is the most beautiful state in Australia?

$P(X=13) + P(X=14) + P(X=15)$

$= {}^{15}C_{13}\left(\frac{5}{6}\right)^{13}\left(\frac{1}{6}\right)^2 + {}^{15}C_{14}\left(\frac{5}{6}\right)^{14}\left(\frac{1}{6}\right) + {}^{15}C_{15}\left(\frac{5}{6}\right)^{15}\left(\frac{1}{6}\right)^0$

$= 105 \times \frac{5^{13}}{6^{15}} + 15 \times \frac{5^{14}}{6^{15}} + 1 \times \frac{5^{15}}{6^{15}}$

$\simeq 0.27260 + 0.19472 + 0.06491$

≈ 0.532 so about 53%

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5 An eight-sided die is inscribed with the digits 1–8.

a What is the probability of obtaining an 8 when the die is thrown?

b Six eight-sided dice are thrown. Construct a table for the distribution of the random variable X that counts the number of eights that occur. Record your results correct to 4 decimal places.

c A player needs to get exactly three eights in order to win. How often would you predict this to occur in 1000 throws of the six dice?

d Repeat part **c** if he needs a throw of three or more eights.

a) $1/8$ b)

x	0	1	2	3	4	5	6
$P(X=x)$	${}^6C_0 \left(\frac{1}{8}\right)^0 \left(\frac{7}{8}\right)^6$ $\simeq 0.4488$	${}^6C_1 \frac{7^5}{8^6}$ $\simeq 0.3847$	${}^6C_2 \frac{7^4}{8^6}$ $\simeq 0.1374$	${}^6C_3 \frac{7^3}{8^6}$ $\simeq 0.0262$	${}^6C_2 \frac{7^2}{8^6}$ $\simeq 0.0028$	${}^6C_1 \frac{7}{8^6}$ $\simeq 0.0002$	${}^6C_0 \frac{1}{8^6}$ $\simeq 0.0000$

c) That would occur approximately $1000 \times 0.0262 \simeq 26$ times.

d) That would occur approximately:

$1000 \times [0.0262 + 0.0028 + 0.0002 + 0.000] \simeq 29$ times.

6 Are the following experiments Bernoulli trials? If so, state the probability of success p and failure q.

- **a** A coin is tossed, and it is noted if the result is heads or tails.
- **b** Two dice are thrown, and the player wins if the sum is more than 10.
- **c** Tests show that 4 out of every 1000 items pass quality control. Consider the random variable 'number of passes' where an item is selected at random from the manufacturing process.
- **d** A card is drawn from a pack, and its suit is noted.

a) Yes : $p = 0.5$ $q = 0.5$

b) Yes, if success is interpreted as the player winning
Success could be 4+6, or 5+5, or 6+4, so 3 favorable outcomes out of 36 possible, i.e $p = \frac{3}{36} = \frac{1}{12}$ $\left(\therefore q = \frac{11}{12}\right)$

c) Yes, with experimental probability $\frac{4}{1000} = 0.004$
Hence $q = 0.996$

d) No, because a Bernouilli trial can have only two outcomes.

7 The binomial distribution $B(n, p)$ consists of n independent Bernoulli trials. Find the mean, variance and standard deviation for each distribution.

a $B(20, 0.2)$ **b** $B(70, 0.5)$ **c** $B(6, 0.8)$

The mean of a binomial distribution $B(n,p)$ is $E(X) = np$

The variance of a $B(n,p)$ is $npq = np(1-p)$

Then $\sigma^2 = Var(X)$, so $\sigma = \sqrt{Var(X)} = \sqrt{npq}$

a) $E(X) = 20 \times 0.2 = 4$

$Var(X) = 20 \times 0.2 \times 0.8 = 3.2$

$\sigma = \sqrt{Var(X)} = \sqrt{3.2} \simeq 1.79$

b) $E(X) = 70 \times 0.5 = 35$

$Var(X) = 70 \times 0.5 \times 0.5 = 17.5$

$\sigma = \sqrt{Var(X)} = \sqrt{17.5} \simeq 4.18$

c) $E(X) = 6 \times 0.8 = 4.8$

$Var(X) = 6 \times 0.8 \times 0.2 = 0.96$

$\sigma = \sqrt{Var(X)} = \sqrt{0.96} \simeq 0.98$

8 A company manufactures mobile phone cases using a mixture of machinery and traditional techniques. Data shows that the probability that a random case will fail quality control is 5%. An inspector selects a random batch of 60 cases from the warehouse. Let X be the binomial random variable of the number of cases that do not pass inspection.

a What is the mean, variance and standard deviation for this distribution?

b Find the probability that the number of cases that fail to pass lies within one standard deviation of the mean.

c New company standards insist that the number of failures in the batch must be no more that one standard deviation above the mean. Batches that fail to meet this standard are rejected. What is the probability of this?

d Due to the new regulations and the number of rejected batches, the company improves its manufacturing process so that the new experimental probability of failure is reduced to 2%. Repeat part **c** to find the new probability that a batch will be rejected.

a) $E(X) = 60 \times 0.05 = 3$

$Var(X) = 60 \times 0.05 \times 0.95 = 2.85$ $\qquad \sigma = \sqrt{Var(X)} = \sqrt{2.85} \simeq 1.69$

b) The mean is 3. That means that "within one standard deviation of the mean" is between $3 - 1.69$ (i.e. $= 1.3$) and $3 + 1.69$ (i.e. $= 4.69$). But X can only take integer values

$P(X=2) + P(X=3) + P(X=4) = {}^{60}C_2\, 0.05^2 \times 0.95^{58} + {}^{60}C_3 \times 0.05^3 \times 0.95^{57} + {}^{60}C_4 \times 0.05^4 \times 0.95^{56}$

$\simeq 0.2259 + 0.2298 + 0.1724 \simeq 0.6281$

c) One standard deviation above the mean is $3 + 1.69 = 4.69$

So we must calculate:

$P(X = 0, 1, 2, 3 \text{ or } 4) = P(X=0) + P(X=1) + P(X = 2, 3 \text{ or } 4)$ calculated at b)

$\simeq {}^{60}C_0 \times 0.05^0 \times 0.95^{60} + {}^{60}C_1 \times 0.05^1 \times 0.95^{59} + 0.6281$

$\simeq 0.0461 + 0.1455 + 0.6281 \simeq 0.8197$

So the probability that a batch is rejected is $1 - 0.8197 = 0.18$ (18%)

d) $E(X) = 60 \times 0.02 = 1.2$ $\qquad Var(X) = 1.2 \times 0.98 = 1.176$ $\qquad \sigma \simeq 1.08$

$P(X = 0, 1 \text{ or } 2) = {}^{60}C_0 \times 0.02^0 \times 0.98^{60} + {}^{60}C_1 \times 0.02 \times 0.98^{59} + {}^{60}C_2 \times 0.02^2 \times 0.98^{58}$

$\simeq 0.2976 + 0.3644 + 0.2194 \simeq 0.8814$

So probability of batch rejected is now $1 - 0.8814 = 12\%$ approx

9 A coin is tossed 80 times.

- **a** Use the exact binomial formula to find the probability of 38, 39 or 40 heads.
- **b** What is the mean and standard deviation for this binomial distribution?
- **c** Calculate np and nq, and state whether this is a situation where a normal approximation may be used.
- **d** In your own words, explain why we calculate $P(37.5 \le X \le 40.5)$ rather than $P(38 \le X \le 40)$.
- **e** Find the probability of 38, 39 or 40 heads using a normal approximation.
- **f** What is the percentage error in this normal approximation?
- **g** Clearly in the example above, there was no need to use an approximation, because the probability could be calculated directly. Calculate now the probability of at least 50 heads using a normal approximation (but do not estimate the percentage error).

a) $P(X=38, 39 \text{ or } 40) = {}^{80}C_{38} \times \frac{1}{2^{80}} + {}^{80}C_{39} \times \frac{1}{2^{80}} + {}^{80}C_{40} \times \frac{1}{2^{80}} \simeq 0.2562$

b) $E(X) = 80 \times \frac{1}{2} = 40$ $\quad Var(X) = 80 \times \frac{1}{2} \times \frac{1}{2} = 20$ $\quad \sigma = \sqrt{20} \simeq 4.47$

c) $np = 40$
$nq = 20$
Both are greater than 5, there approximation using a Normal distribution is possible.

d) The Continuity correction must be applied when we approximate the Binomial distribution (discrete) by a Normal distribution (continuous curve)

e) $P(37.5 \leqslant X \leqslant 40.5) = P\left(\frac{37.5-40}{\sqrt{20}} < z < \frac{40.5-40}{\sqrt{20}}\right)$

$= P(-0.5590 < z < 0.1118)$

$= P(z < 0.1118) - P(z < -0.5590)$

$\simeq 0.54451 - 0.28808 \simeq 0.2564$ so 25.64% approx

f) $\% \text{ error} = \frac{0.2564 - 0.2562}{0.2564} \simeq 0.0008$ so less than 1%

g) $P(X \geqslant 49.5) = P\left(z > \frac{49.5-40}{\sqrt{20}}\right) = P(z > 2.1243)$

$= 1 - P(z < 2.1243) = 1 - 0.98318 = 0.01682$

so 1.7% approx

10 The sum of two dice is recorded.

a Find the probability that the sum is at least 10.

b Use a normal approximation to find the probability that the sum is at least 10 in more than 14 out of 80 throws.

a)

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

There are six favorable events, hence $P(\text{sum} \geq 10) = \frac{6}{36} = \frac{1}{6}$

b) $X \sim B(80, 1/6)$

which can be approximated by $N\left(80\times\frac{1}{6}, 80\times\frac{1}{6}\times\frac{5}{6}\right)$

i.e. by $N\left(40/3, 100/9\right)$

$P(X \geq 15) = P\left(Z \geq \frac{14.5 - 40/3}{\sqrt{100/9}}\right) = P\left(Z \geq \frac{7/6}{10/3}\right) = P(Z \geq 0.35)$

$= 1 - P(Z < 0.35)$

$= 1 - 0.6368 \simeq 0.363$

so 36% approximately

12 A bag contains 3 red balls and 2 white balls. Five balls are selected in turn, with replacement, and the number of red balls is recorded.

a Construct a table showing the theoretical sample proportions of red balls that are selected.

b Find the probability that the proportion of red balls is less than:

i 40%, **ii** 50%.

c Find the mean, variance and standard deviation for the random variable $\hat{p}$ tabulated in part **a**.

a)

$\hat{p}$	0	0.2	0.4	0.6	0.8	1
$P(\hat{p})$	${}^5C_0\left(\frac{3}{5}\right)^0\left(\frac{2}{5}\right)^5$ $=\frac{32}{3125}$ ≈ 0.0102	${}^5C_1\,\frac{3\times2^4}{5^5}$ $=\frac{48}{625}$ ≈ 0.0768	${}^5C_2\,\frac{3^2 2^3}{5^5}$ $=\frac{144}{625}$ $=0.2304$	${}^5C_3\,\frac{3^3 2^2}{5^5}$ $=\frac{216}{625}$ $=0.3456$	${}^5C_4\,\frac{3^4\times 2}{5^5}$ $=\frac{162}{625}$ $=0.2592$	${}^5C_5\,\frac{3^5}{5^5}$ $=\frac{243}{3125}$ ≈ 0.0778

b) i) $P(\hat{p}<0.4) = P(\hat{p}=0) + P(\hat{p}=0.2)$

$$= \frac{32}{3125} + \frac{48}{625} = \frac{272}{3125} = 0.0870$$

ii) $P(\hat{p}<0.5) = \underbrace{P(\hat{p}=0) + P(\hat{p}=0.2)}_{=\frac{272}{3125}} + P(\hat{p}=0.4)$

$$= \frac{272}{3125} + \frac{144}{625} = \frac{992}{3125} \simeq 0.3174$$

c) Mean $= \hat{p} = \frac{3}{5}$

$$\text{Variance} = \frac{pq}{n} = \frac{3/5 \times 2/5}{5} = \frac{6/25}{5} = \frac{6}{125}$$

$$\sigma = \sqrt{\text{Variance}} = \sqrt{\frac{6}{125}} = \frac{1}{5}\sqrt{\frac{6}{5}} = \frac{\sqrt{30}}{25}$$

13 A fair die is to be thrown 500 times.

a Find the mean and standard deviation for the sample proportion of sixes in the theoretical distribution for this experiment.

b In one sample of 500 throws, the number of sixes was 70. How many standard deviations is this result below the mean?

a) $\mu = p = \frac{1}{6}$

$$\text{Variance} = \frac{pq}{n} = \frac{1/6 \times 5/6}{500} = \frac{5}{18{,}000} = \frac{1}{3600}$$

$$\sigma = \sqrt{\text{Variance}} = \sqrt{\frac{1}{3600}} = \frac{1}{60}$$

b) The z-score is $\dfrac{\frac{70}{500} - \frac{1}{6}}{\frac{1}{60}} = \dfrac{-\frac{2}{75}}{\frac{1}{60}} = -1.6$

so it's 1.6 standard deviation below the mean.

14 Long-term records show that the percentage of male babies born in a large hospital is 53%. A study is carried out on the effect of a high potassium diet (white beans, salmon, avocados, almonds, apples and mushrooms) on increasing the probability that the baby will be male. In the group of 653 births under this study, with the mother following this diet, more than 54% were male. What is the probability of this happening by chance? Use a normal approximation for the sample proportion with no continuity correction.

$$P(X > 0.53) = P\left(z > \frac{0.54 - 0.53}{\sqrt{\frac{0.53 \times 0.47}{653}}}\right)$$

$$= P\left(z > \frac{0.01}{0.01953}\right)$$

$$= P(z > 0.512)$$

$$= 1 - P(z < 0.512)$$

$$= 1 - 0.6957$$

$$= 0.304$$

so approx 30.4%