DEFINITE INTEGRALS AND SUBSTITUTION

1 Evaluate: **(a)** $\int_0^1 x\sqrt{1-x^2}\,dx$ using the substitution $u = 1 - x^2$

(b) $\int_{-1}^2 x\sqrt{2-x}\,dx$ using the substitution $u = 2 - x$

a) $u = 1 - x^2$ so $\frac{du}{dx} = -2x$ or $x\,dx = -\frac{1}{2}\,du$

when $x = 0$ $u = 1$ when $x = 1$, $u = 0$, So

$$\int_0^1 x\sqrt{1-x^2}\,dx = \int_1^0 \sqrt{u} \times \left(-\frac{1}{2}\,du\right) = \frac{1}{2}\int_0^1 \sqrt{u}\,du = \frac{1}{2}\int_0^1 u^{1/2}\,du$$

$$= \frac{1}{2}\left[\frac{u^{1/2+1}}{\frac{1}{2}+1}\right]_0^1 = \frac{1}{2}\left[\frac{u^{3/2}}{\frac{3}{2}}\right]_0^1 = \frac{1}{3}\left[u^{1/3}\right]_0^1 = \frac{1}{3}$$

b) $u = 2 - x$ so $\frac{du}{dx} = -1$ or $du = -dx$

$x = 2 - u$ $dx = -du$

when $x = -1$, $u = 3$ and when $x = 2$, $u = 0$, so

$$\int_{-1}^2 x\sqrt{2-x}\,dx = \int_3^0 (2-u)\sqrt{u}\,(-du) = \int_0^3 (2-u)\sqrt{u}\,du$$

$$= 2\int_0^3 \sqrt{u}\,du - \int_0^3 u^{3/2}\,du$$

$$= 2\int_0^3 u^{1/2}\,du - \left[\frac{u^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right]_0^3$$

$$= 2\left[\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_0^3 - \left[\frac{u^{5/2}}{\frac{5}{2}}\right]_0^3$$

$$= 2\left[\frac{u^{3/2}}{\frac{3}{2}}\right]_0^3 - \frac{2}{5}\left[u^{5/2}\right]_0^3 = \frac{4}{3}\left[u^{3/2}\right]_0^3 - \frac{2}{5}\left[u^{5/2}\right]_0^3$$

$$= \frac{4}{3}\times 3^{3/2} - \frac{2}{5}\times 3^{5/2} = 4\sqrt{3} - \frac{2}{5}\times 3^{5/2} = \sqrt{3}\left[4 - \frac{18}{5}\right]$$

$$= \frac{2\sqrt{3}}{5}$$

DEFINITE INTEGRALS AND SUBSTITUTION

4 Evaluate: **(a)** $\int_3^4 (2x-3)(x^2-3x+2)^2\,dx$ using the substitution $u = x^2 - 3x + 2$

(b) $\int_0^2 \frac{x}{(x^2+2)^2}\,dx$ using the substitution $u = x^2 + 2$

a) $u = x^2 - 3x + 2$ so $\frac{du}{dx} = 2x - 3$ and $du = (2x-3)\,dx$

when $x = 3$ $u = 3^2 - 3 \times 3 + 2 = 2$

when $x = 4$ $u = 4^2 - 3 \times 4 + 2 = 6$, So

$$\int_3^4 (2x-3)(x^2-3x+2)^2\,dx = \int_2^6 u^2\,du = \left[\frac{u^3}{3}\right]_2^6 = \frac{6^3}{3} - \frac{2^3}{3} = 72 - \frac{8}{3} = 69\tfrac{1}{3}$$

b) $u = x^2 + 2$ so $\frac{du}{dx} = 2x$ $du = 2x\,dx$ or $x\,dx = \frac{du}{2}$

when $x = 0$, $u = 2$; when $x = 2$, $u = 2^2 + 2 = 6$, so

$$\int_0^2 \frac{x}{(x^2+2)^2}\,dx = \int_2^6 \frac{1}{u^2} \times \frac{1}{2}\,du = \frac{1}{2}\int_2^6 \frac{du}{u^2} = \frac{1}{2}\int_2^6 u^{-2}\,du$$

$$= \frac{1}{2}\left[\frac{u^{-2+1}}{-2+1}\right]_2^6 = \frac{1}{2}\left[\frac{u^{-1}}{-1}\right]_2^6 = -\frac{1}{2}\left[\frac{1}{u}\right]_2^6$$

$$= \frac{1}{2}\left[\frac{1}{u}\right]_6^2 = \frac{1}{2}\left[\frac{1}{2} - \frac{1}{6}\right] = \frac{1}{6}$$

DEFINITE INTEGRALS AND SUBSTITUTION

5 Evaluate: **(a)** $\int_0^1 \frac{t}{\sqrt{1+t}}\,dt$ using the substitution $u = 1 + t$

(b) $\int_0^1 3x^2\left(x^3-1\right)^4 dx$ using the substitution $u = x^3 - 1$

a) $u = 1 + t$ $\quad \frac{du}{dt} = 1$ $\quad$ so $du = dt$

$t = u - 1$

when $t = 0$, $u = 1$; when $t = 1$, $u = 2$, so

$$\int_0^1 \frac{t}{\sqrt{1+t}}\,dt = \int_1^2 \frac{(u-1)}{\sqrt{u}}\,du = \int_1^2 \left(\sqrt{u} - u^{-1/2}\right) du = \int_1^2 \left(u^{1/2} - u^{-1/2}\right) du$$

$$= \left[\frac{u^{\frac{1}{2}+1}}{\frac{3}{2}} - \frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]_1^2 = \left[\frac{2u^{3/2}}{3} - 2u^{1/2}\right]_1^2 = 2\left[\sqrt{u}\left(\frac{u}{3} - 1\right)\right]_1^2$$

$$= 2\left[\sqrt{2}\left(\frac{2}{3} - 1\right) - 1\left(\frac{1}{3} - 1\right)\right] = 2\left[-\frac{\sqrt{2}}{3} + \frac{2}{3}\right] = \frac{4 - 2\sqrt{2}}{3}$$

b) $u = x^3 - 1$ $\quad \frac{du}{dx} = 3x^2$ $\quad$ so $du = 3x^2\,dx.$

when $x = 0$, $u = -1$; when $x = 1$, $u = 0$; So

$$\int_0^1 3x^2\left(x^3-1\right)^4 dx = \int_{-1}^0 u^4\,du = \left[\frac{u^5}{5}\right]_{-1}^0$$

$$= \frac{0^5}{5} - \left(\frac{(-1)^5}{5}\right) = \frac{1}{5}$$

10 Find the area of the region bounded by the curve $y=\frac{x}{\sqrt{x^2-1}}$, the x-axis and the lines $x=\sqrt{2}$ and $x=\sqrt{5}$.

For $x=\sqrt{2}$ $\quad y=\frac{\sqrt{2}}{\sqrt{2-1}}=\sqrt{2}\approx 1.4$

For $x=\sqrt{5}$ $\quad y=\frac{\sqrt{5}}{\sqrt{5-1}}=\frac{\sqrt{5}}{2}\approx 1.18$

The curve is always above the x-axis between $x=\sqrt{2}$ and $x=\sqrt{5}$, therefore this area is given by $\int_{\sqrt{2}}^{\sqrt{5}}\frac{x}{\sqrt{x^2-1}}\,dx$

$u=x^2-1 \qquad \frac{du}{dx}=2x \qquad$ so $du=2x\,dx$

or $x\,dx=\frac{1}{2}du$.

when $x=\sqrt{2}$, $u=1$; and when $x=\sqrt{5}$, $u=5-1=4$; SO

$$\int_{\sqrt{2}}^{\sqrt{5}}\frac{x}{\sqrt{x^2-1}}\,dx=\int_1^4\frac{1}{\sqrt{u}}\times\frac{1}{2}\,du=\frac{1}{2}\int_1^4 u^{-1/2}\,du$$

$$=\frac{1}{2}\left[\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]_1^4=\frac{1}{2}\left[\frac{u^{1/2}}{\frac{1}{2}}\right]_1^4=\left[u^{1/2}\right]_1^4$$

$$=\left[\sqrt{u}\right]_1^4=\sqrt{4}-\sqrt{1}=2-1=1\text{ unit}^2$$

For information: