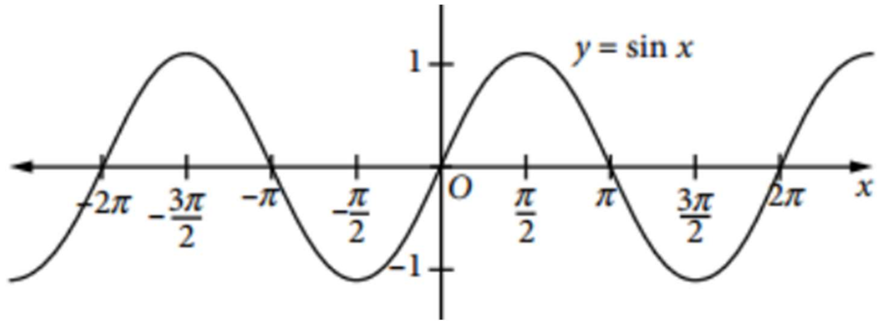
THE INVERSE SINE FUNCTION

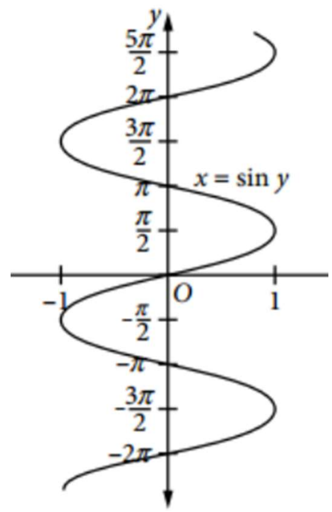
The graph of $f(x) = sin\, x$ is shown below.



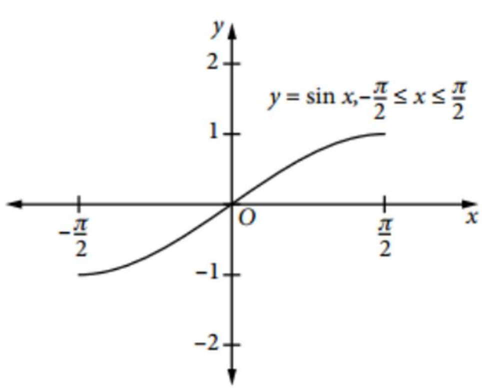
As every y between -1 and 1 have multiple possible values for x, it is a many-to-one function.

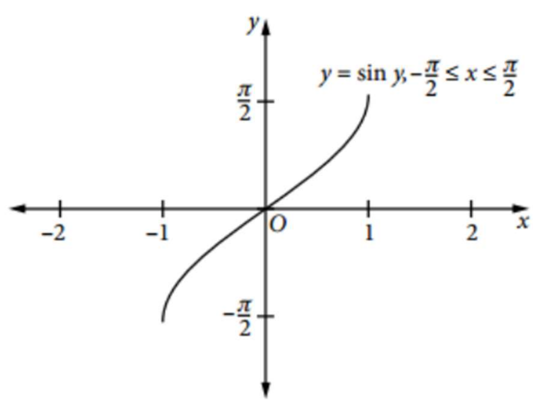
The reflection of $f(x) = sin\, x$ in the line $y = x$ is shown to the right.

It is not a function as for each x, there are many possible values of y (i.e. "it doesn't pass the vertical line test").



By restricting the domain of $f(x) = sin\, x$ to $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, the function f becomes one-to-one, as shown on the graph below left. Its reflection in the line $y = x$ is shown below right: this is the graph of the inverse function of $f(x) = sin\, x$ (for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$)





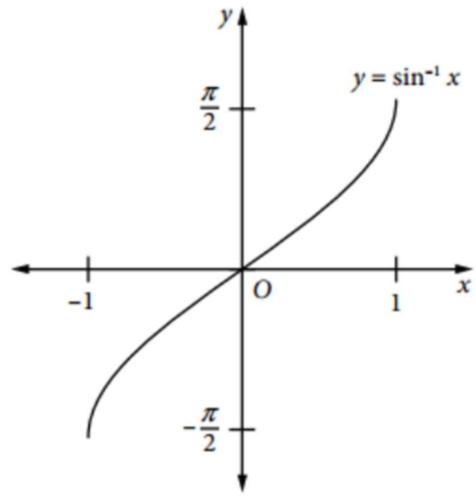
The inverse function of $f(x) = sin\, x$ (for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$) is noted $sin^{-1}\, x$ or $arcsin\, x$; it is only defined for $-1 \le x \le 1$.

So $f^{-1}(x) = sin^{-1}\, x$ (also noted $f^{-1}(x) = arcsin\, x$) only exists for $-1 \le x \le 1$.

INVERSE TRIGONOMETRIC FUNCTIONS

FEATURES OF THE INVERSE SINE FUNCTION:

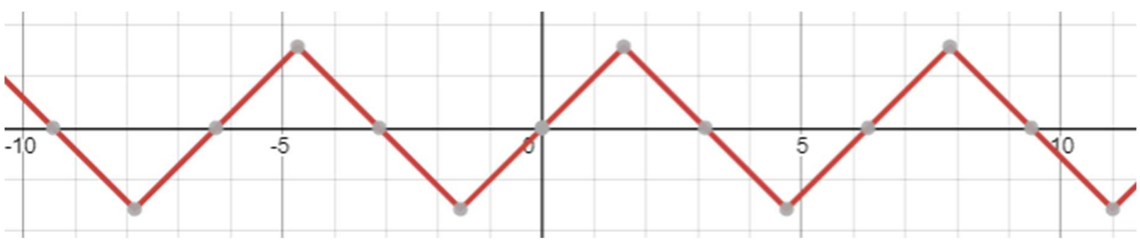
- **increasing** function
- **domain** is $-1 \leq x \leq 1$
- **range** is $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
- it has **vertical tangents at its endpoints**
- it is symmetrical with regard to the origin O, so it is an **odd function** [for all x in the domain, $f(-x) = -f(x)$, or in that case $sin^{-1}(-x) = -sin^{-1}(x)$]



Note that:

1. $y = sin(sin^{-1} x)$ only exists for values of x between (-1) and $(+1)$ inclusive; its graph is the same than $y = x$ on the entire domain of the function (i.e. $-1 \leq x \leq 1$).

2. $y = sin^{-1}(sin\, x)$ exists for all values of x, however its graph is the same than $y = x$ only for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ (otherwise it has a sawtooth shape, not studied in detail in this course).



Example 7

Find the exact values of the following.

(a) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ **(b)** $\sin^{-1}\left(-\frac{1}{2}\right)$ **(c)** arcsin 1.2

(d) $\sin^{-1}(\sin 1.2)$ **(e)** $\arcsin\left(\sin\frac{\pi}{4}\right)$ **(f)** $\sin^{-1}(\sin\pi)$

Solution

(a) As $-1 \leq \frac{\sqrt{3}}{2} \leq 1$, $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ exists. It is the number y or angle y^c (i.e. in radians), such that $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ and whose sine is $\frac{\sqrt{3}}{2}$. Hence $\sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$.

(b) Similarly, $\sin^{-1}\left(-\frac{1}{2}\right)$ can be evaluated as a number y or angle y^c, such that $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, whose sine is $-\frac{1}{2}$. Hence $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

Alternatively: $\sin^{-1} x$ is an odd function, so $\sin^{-1}\left(-\frac{1}{2}\right) = -\sin^{-1}\left(\frac{1}{2}\right) = -\frac{\pi}{6}$.

(c) 1.2 is not within the domain $-1 \leq x \leq 1$, so arcsin 1.2 does not exist.

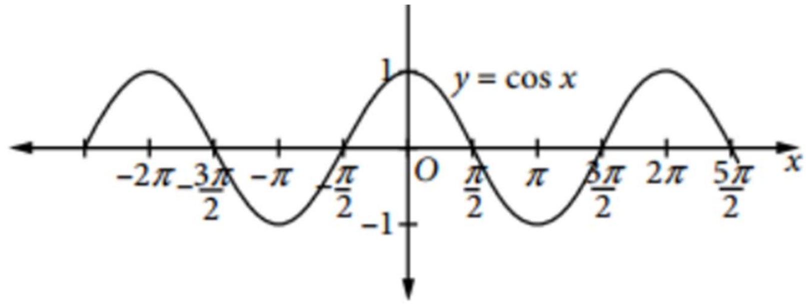
(d) 1.2 is within the domain $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, so $\sin^{-1}(\sin 1.2) = 1.2$.

(e) $\frac{\pi}{4}$ is within the domain $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, so $\arcsin\left(\sin\frac{\pi}{4}\right) = \frac{\pi}{4}$.

(f) π is outside the domain $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, so $\sin^{-1}(\sin\pi) \neq \pi$. Instead, $\sin^{-1}(\sin\pi) = \sin^{-1} 0 = 0$.

THE INVERSE COSINE FUNCTION

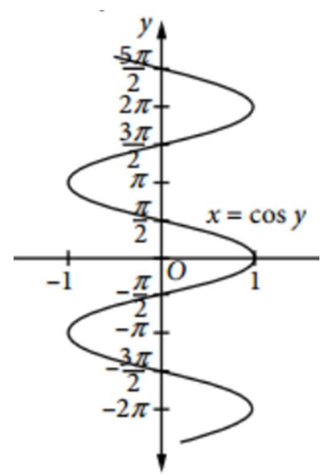
The graph of $f(x) = cos\,x$ is shown below.



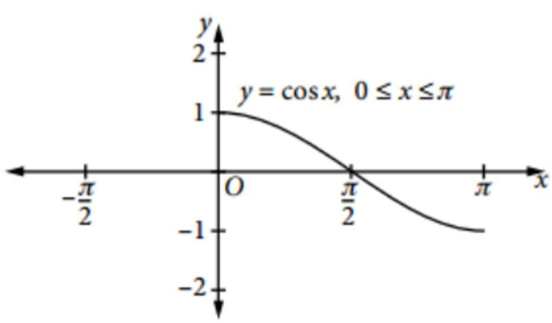
As every y between -1 and 1 have multiple possible values for x, it is a **many-to-one** function.

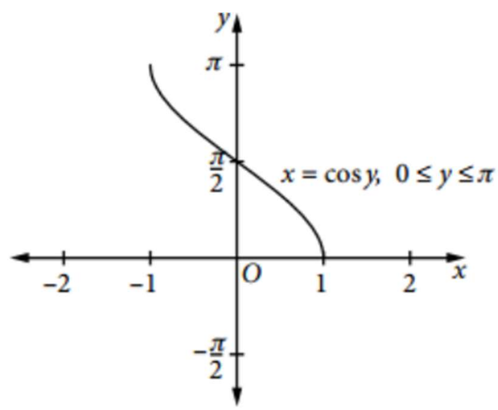
The reflection of $f(x) = cos\,x$ in the line $y = x$ is shown to the right.

It is not a function as for each x, there are many possible values of y (i.e. "it doesn't pass the vertical line test").



By restricting the domain of $f(x) = cos\,x$ to $0 \leq x \leq \pi$, the function f becomes one-to-one, as shown on the graph below left. Its reflection in the line $y = x$ is shown below right: this is the graph of the inverse function of $f(x) = cos\,x$ (for $0 \leq x \leq \pi$)

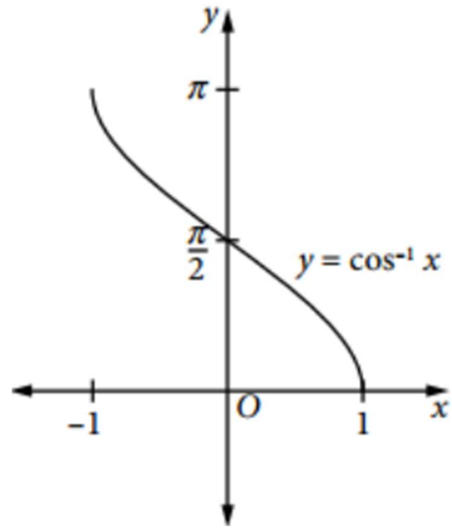




The inverse function of $f(x) = cos\,x$ (for $0 \leq x \leq \pi$) is noted $cos^{-1}\,x$ or $arccos\,x$; it is only defined for $-1 \leq x \leq 1$.

So $f^{-1}(x) = cos^{-1}\,x$ (also noted $f^{-1}(x) = arccos\,x$) only exists for $-1 \leq x \leq 1$.

INVERSE TRIGONOMETRIC FUNCTIONS



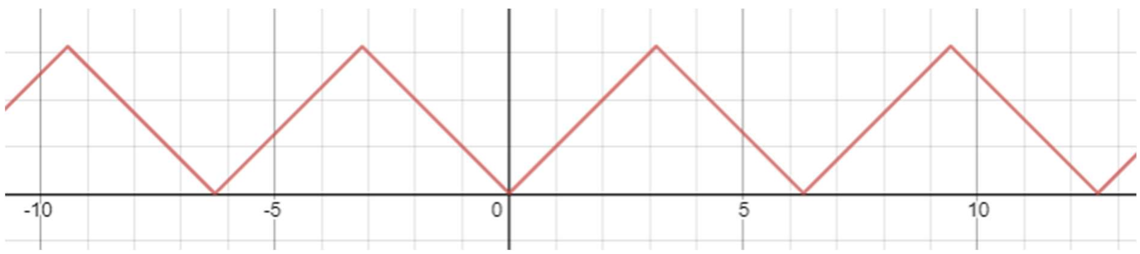
FEATURES OF THE INVERSE COSINE FUNCTION:

- **decreasing** function
- **domain** is $-1 \leq x \leq 1$
- **range** is $0 \leq y \leq \pi$
- it has **vertical tangents at its endpoints**
- the function is neither odd nor even, however it does have a **rotational symmetry about its *y*-intercept.** So for any x in the domain, the sum of the functions heights at $(-x)$ and $(+x)$ will always be π, i.e.: $\cos^{-1}(-x) + \cos^{-1}(x) = \pi$

Note that:

1. $y = cos(cos^{-1} x)$ only exists for values of x between (-1) and $(+1)$ inclusive; its graph is the same than $y = x$ on the entire domain of the function (i.e. $-1 \leq x \leq 1$).

2. $y = cos^{-1}(cos\, x)$ exists for all values of x, however its graph is the same than $y = x$ only for $0 \leq x \leq \pi$ (otherwise it has a sawtooth shape, not studied in detail in this course).



Example 9

Find the exact values of the following. **(a)** $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ **(b)** $\cos\left(\arccos\left(-\frac{1}{2}\right)\right)$ **(c)** $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)$ **(d)** $\arccos\left(\cos\frac{5\pi}{3}\right)$ **(e)** $\sin\left(\cos^{-1}\left(-\frac{1}{2}\right)\right)$ **(f)** $\tan\left(\arccos\left(-\frac{2}{3}\right)\right)$

Solution

(a) Method 1

Let $y = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Then $\cos y = -\frac{1}{\sqrt{2}}$ and $0 \leq y \leq \pi$

$\therefore\ y = \frac{3\pi}{4}$

$\therefore\ \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$

Method 2

$$\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \pi - \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

(b) $\cos(\arccos x) = x$ for $-1 \leq x \leq 1$, so $\cos\left(\arccos\left(-\frac{1}{2}\right)\right) = -\frac{1}{2}$

(c) $\cos^{-1}(\cos x) = x$ for $0 \leq x \leq \pi$, so $\cos^{-1}\left(\cos\frac{2\pi}{3}\right) = \frac{2\pi}{3}$

(d) $\frac{5\pi}{3}$ is not in the domain $0 \le x \le \pi$, so $\arccos\left(\cos\frac{5\pi}{3}\right) \neq \frac{5\pi}{3}$

The solution is: $\arccos\left(\cos\frac{5\pi}{3}\right) = \arccos\left(\cos\frac{\pi}{3}\right) = \frac{\pi}{3}$

(e) **Method 1**

$$\sin\left(\cos^{-1}\left(-\tfrac{1}{2}\right)\right) = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$$

Method 2

$$\begin{aligned}\sin\left(\cos^{-1}\left(-\tfrac{1}{2}\right)\right) &= \sin\left(\pi - \cos^{-1}\tfrac{1}{2}\right)\\ &= \sin\left(\pi - \tfrac{\pi}{3}\right)\\ &= \sin\frac{2\pi}{3}\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

Method 2 here shows a good approach. When you have to take an inverse trigonometric function of a negative value, use the symmetry properties of the inverse trigonometric functions:

$\sin^{-1}(-x) = -\sin^{-1}x$ $\quad\cos^{-1}(-x) = \pi - \cos^{-1}x$ $\quad\tan^{-1}(-x) = -\tan^{-1}x$

This process ensures that the function is evaluated with a first quadrant angle.

(f) **Method 1**

Let $\arccos\left(-\frac{2}{3}\right) = \theta$

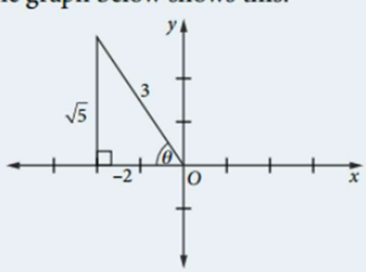
Then $\cos\theta = -\frac{2}{3}$ and $0 \le \theta \le \pi$

So θ is a second quadrant angle.

Need to evaluate:

$\tan\left(\arccos\left(-\frac{2}{3}\right)\right) = \tan\theta$

The graph below shows this:



θ is in the second quadrant, $\cos\theta = -\frac{2}{3}$

Need to find the value of $\tan\theta$.

$\therefore\ \tan\left(\arccos\left(-\frac{2}{3}\right)\right) = \tan\theta = -\frac{\sqrt{5}}{2}$

Method 2

Use the symmetry properties:

$\arccos\left(-\frac{2}{3}\right) = \pi - \arccos^{-1}\left(\frac{2}{3}\right)$ and $\tan(\pi - \theta) = -\tan\theta$

$$\begin{aligned}\tan\left(\arccos\left(-\tfrac{2}{3}\right)\right) &= \tan\left(\pi - \arccos\tfrac{2}{3}\right)\\ &= -\tan\left(\arccos\tfrac{2}{3}\right)\end{aligned}$$

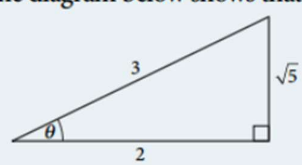
Now find the exact value of $\tan\left(\arccos\frac{2}{3}\right)$.

Let $\theta = \arccos\frac{2}{3}$

$\therefore\ \cos\theta = \frac{2}{3}$ (where θ is acute)

Now evaluate $\tan\left(\arccos\frac{2}{3}\right) = \tan\theta$.

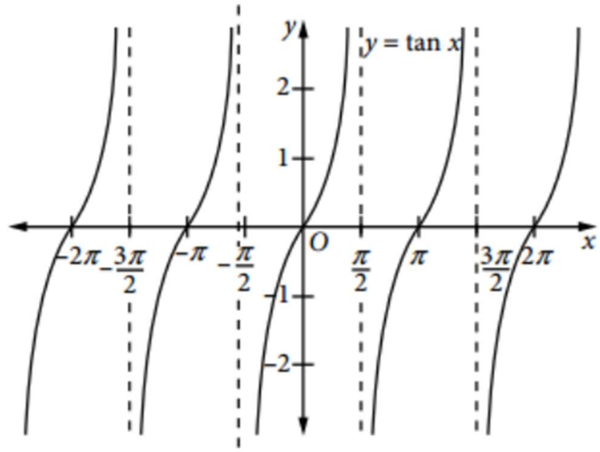
The diagram below shows that if $\cos\theta = \frac{2}{3}$ then $\tan\theta = \frac{\sqrt{5}}{2}$:



Hence $\tan\left(\arccos\left(-\frac{2}{3}\right)\right) = -\frac{\sqrt{5}}{2}$.

THE INVERSE TANGENT FUNCTION

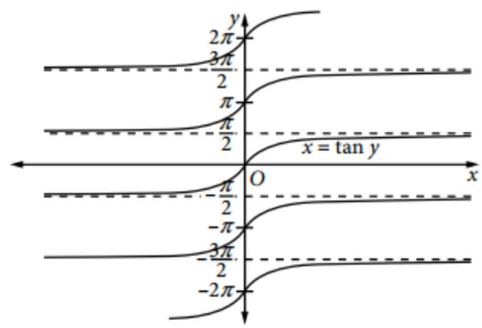
The graph of $f(x) = tan\, x$ is shown below.



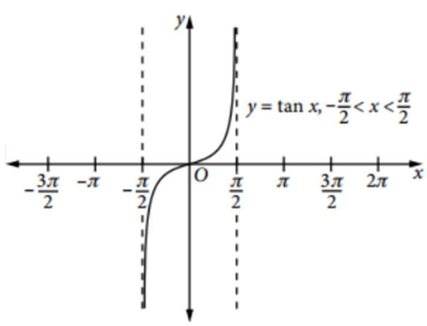
As every y have multiple possible values for x, it is a **many-to-one** function.

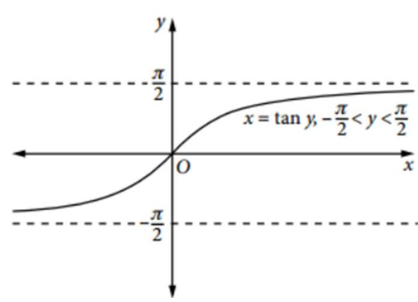
The reflection of $f(x) = tan\, x$ in the line $y = x$ is shown to the right.

It is not a function as for each x, there are many possible values of y (i.e. "it doesn't pass the vertical line test").



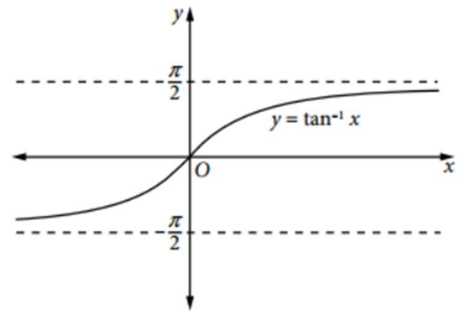
By restricting the domain of $f(x) = tan\, x$ to $-\frac{\pi}{2} < x < \frac{\pi}{2}$, the function f becomes one-to-one, as shown on the graph below left. Its reflection in the line $y = x$ is shown below right: this is the graph of the inverse function of $f(x) = tan\, x$ (for $-\frac{\pi}{2} < x < \frac{\pi}{2}$)





The inverse function of $f(x) = tan\, x$ (for $-\frac{\pi}{2} < x < \frac{\pi}{2}$) is noted $f^{-1}(x) = tan^{-1}\, x$ or $f^{-1}(x) = arctan\, x$).

FEATURES OF THE INVERSE TANGENT FUNCTION:

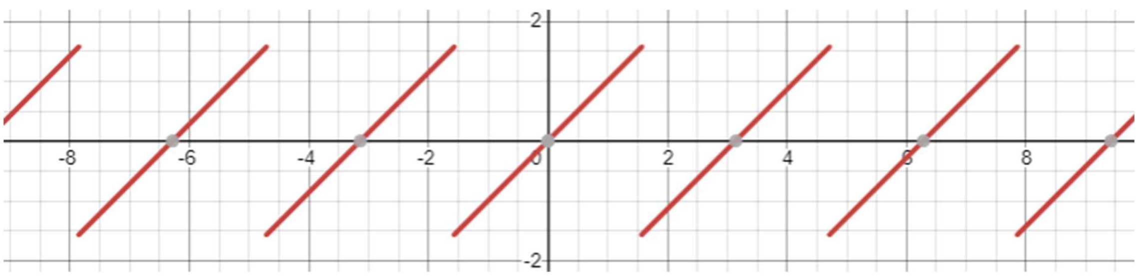


- **increasing** function
- **domain** is $\mathbb{R}$
- **range** is $-\frac{\pi}{2} < y < \frac{\pi}{2}$ (not inclusive)
- it has horizontal asymptotes $y = \pm\frac{\pi}{2}$
- it is symmetrical with regard to the origin O, so it is an **odd function** [for all x in the domain, $f(-x) = -f(x)$, or in that case $tan^{-1}(-x) = -tan^{-1}(x)$]

Note that:

3. $y = tan(tan^{-1} x)$ is the same than $y = x$.

4. $y = tan^{-1}(tan\, x)$ exists everywhere except for $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$. Its graph is the same to $y = x$ only for $-\frac{\pi}{2} < y < \frac{\pi}{2}$ (otherwise it looks like an infinite set of parallel intervals with open circles on each end, not studied in detail in this course).



Example 11

Find the exact values of the following.

(a) $\arctan\left(\frac{1}{\sqrt{3}}\right)$ **(b)** $\tan(\tan^{-1} 1)$ **(c)** $\arctan\left(\tan\left(\frac{\pi}{3}\right)\right)$ **(d)** $\tan^{-1}\left(\tan\left(-\frac{4\pi}{3}\right)\right)$

Solution

(a) $\arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$. It is the value of θ, between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (not inclusive), for which $\tan\theta = \frac{1}{\sqrt{3}}$.

(b) $\tan(\tan^{-1} 1) = 1$ **(c)** $\arctan\left(\tan\left(\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$, as $\frac{\pi}{3}$ is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (not inclusive).

(d) $\tan^{-1}\left(\tan\left(-\frac{4\pi}{3}\right)\right)$ is not equal to $-\frac{4\pi}{3}$, because $-\frac{4\pi}{3}$ is not between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

Using the symmetry properties that $\tan\theta$ and $\tan^{-1} x$ are odd functions, and that $\tan(\pi + \theta) = \tan\theta$:

$\tan^{-1}\left(\tan\left(-\frac{4\pi}{3}\right)\right) = \tan^{-1}\left(-\tan\left(\frac{4\pi}{3}\right)\right) = -\tan^{-1}\left(\tan\left(\frac{4\pi}{3}\right)\right) = -\tan^{-1}\left(\tan\left(\frac{\pi}{3}\right)\right) = -\frac{\pi}{3}$

Example 12

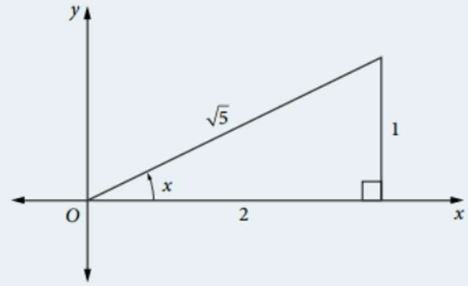
Find the exact value of $\sin\left(2\tan^{-1}\frac{1}{2}\right)$.

Solution

Let $\tan^{-1}\frac{1}{2}=x$

Thus $\tan x=\frac{1}{2}$ and $-\frac{\pi}{2}<x<\frac{\pi}{2}$

Hence x can be represented as a first quadrant angle.



Then: $\sin\left(2\tan^{-1}\frac{1}{2}\right)=\sin 2x$

$$=2\sin x\cos x$$

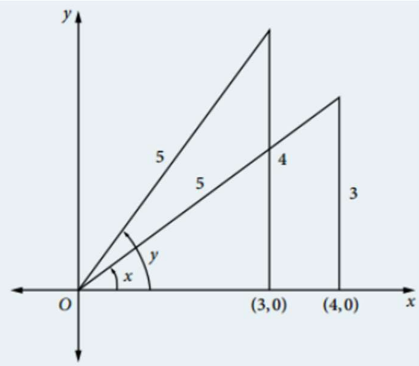
$$=2\left(\frac{1}{\sqrt{5}}\right)\left(\frac{2}{\sqrt{5}}\right)$$

$$=\frac{4}{5}$$

Example 13

Find $\sin\left[\cos^{-1}\frac{4}{5}+\tan^{-1}\left(-\frac{4}{3}\right)\right]$.

Solution



$\sin\left[\cos^{-1}\frac{4}{5}+\tan^{-1}\left(-\frac{4}{3}\right)\right]=\sin\left[\cos^{-1}\frac{4}{5}-\tan^{-1}\frac{4}{3}\right]$

$=\sin(x-y)$ where $x=\cos^{-1}\frac{4}{5}$ and $y=\tan^{-1}\frac{4}{3}$

$=\sin\left(\cos^{-1}\frac{4}{5}\right)\cos\left(\tan^{-1}\frac{4}{3}\right)-\cos\left(\cos^{-1}\frac{4}{5}\right)\sin\left(\tan^{-1}\frac{4}{3}\right)$

Using expansion of $\sin(x-y)$:

$\cos^{-1}\frac{4}{5}=x$, so $\cos x=\frac{4}{5}$ and $0\le x\le\pi$

$\tan^{-1}\frac{4}{3}=y$, so $\tan y=\frac{4}{3}$ and $-\frac{\pi}{2}<y<\frac{\pi}{2}$

Hence both x and y can be represented as first quadrant angles:

$$\sin\left[\cos^{-1}\frac{4}{5}+\tan^{-1}\left(-\frac{4}{3}\right)\right]=\sin x\cos y-\cos x\sin y$$

$$=\left(\frac{3}{5}\right)\left(\frac{3}{5}\right)-\left(\frac{4}{5}\right)\left(\frac{4}{5}\right)$$

$$=-\frac{7}{25}$$

Example 14

Prove that $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$ for $-1\le x\le 1$.

Solution

Let $\alpha=\sin^{-1}x$ $\quad\therefore\ \sin\alpha=x$ where $-\frac{\pi}{2}\le\alpha\le\frac{\pi}{2}$

Recall that $\cos\left(\frac{\pi}{2}-\alpha\right)=\sin\alpha$, so $\cos\left(\frac{\pi}{2}-\alpha\right)=x$

Also, as $-\frac{\pi}{2}\le\alpha\le\frac{\pi}{2}$, thus $0\le\frac{\pi}{2}-\alpha\le\pi$

$\therefore\ \frac{\pi}{2}-\alpha=\cos^{-1}x$ (noting that $\theta=\cos^{-1}x$ only when $\cos\theta=x$ and $0\le\theta\le\pi$)

$\therefore\ \frac{\pi}{2}-\sin^{-1}x=\cos^{-1}x$ so $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$

You should remember this result: $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$ for $-1\le x\le 1$