SIMPLE HARMONIC MOTION (SHM)

1 The displacement x m of a particle moving in a straight line is given by $x = 6\cos 4t$. Describe the motion of the particle.

$x(t) = 6 \cos 4t$

$\dot{x}(t) = 6 \times 4 \,(-\sin 4t) = -24 \sin 4t$

$\ddot{x}(t) = -24 \times 4 \cos 4t = -96 \cos 4t$

So the particle satisfies the equation $\ddot{x} = -16x$, hence it is in a simple harmonic motion (as the acceleration is proportional to the displacement).

In fact, we could have stated that from the equation of the displacement $x = 6\cos 4t$, which is of the form $x = a\cos(nt + \alpha)$, so the motion is SHM about the origin, with $a = 6$ and $n = 4$, and $\alpha = 0$.

Period is $\dfrac{2\pi}{4} = \dfrac{\pi}{2}$

The particle starts 6 cm to the right of O

At $t = 0$ $\quad \dot{x}(0) = -24 \sin(4 \times 0) = 0$ (so the particle is initially at rest).

At $t = 0$ $\quad \ddot{x}(0) = -96 \cos(4 \times 0) = -96 \text{ m s}^{-2}$

SIMPLE HARMONIC MOTION (SHM)

2 The equation of motion of a particle moving with simple harmonic motion is $\ddot{x} = -9x$. Find its period, amplitude and greatest speed if: **(a)** $x = 0$, $\dot{x} = 2$ when $t = 0$ **(b)** $x = 2$, $\dot{x} = 2$ when $t = 0$.

The equation of motion is in the form $\ddot{x} = -9x$, therefore $n = 3$ and $C = 0$. So $x(t) = a \sin(3t + \alpha)$.

Period is $T = \frac{2\pi}{n} = \frac{2\pi}{3}$

a) At $t = 0$, $x = 0$ so $0 = a \sin \alpha$

so $\alpha = 0$, i.e. $x(t) = a \sin(3t)$.

At $t = 0$, $\dot{x} = 2$ so $\dot{x}(t) = 3a \cos(3t)$

i.e. $2 = 3a \times \cos 0$ so $a = 2/3$

The greatest speed is when $\cos(3t) = 1$, i.e. $\dot{x} = 3 \times \frac{2}{3} = 2\ ms^{-1}$

b) At $t = 0$, $x = 2$ so $2 = a \sin \alpha$

At $t = 0$, $\dot{x} = 2$ so $2 = 3a \cos(3 \times 0 + \alpha) = 3a \cos \alpha$

so $\sin \alpha = \frac{2}{a}$ Hence $\sin^2 \alpha + \cos^2 \alpha = \left(\frac{2}{a}\right)^2 + \left(\frac{2}{3a}\right)^2$

and $\cos \alpha = \frac{2}{3a}$ $\Leftrightarrow 1 = \frac{4}{a^2} + \frac{4}{9a^2}$

so $a^2 = 4 + \frac{4}{9} = \frac{40}{9}$ so $a = \frac{\sqrt{40}}{3} = \frac{2\sqrt{10}}{3}$

$\dot{x}(t) = 3 \times \frac{2\sqrt{10}}{3} \cos(3t + \alpha) = 2\sqrt{10} \cos(3t + \alpha)$

The speed is greatest when $\cos(3t + \alpha) = 1$, i.e. $\dot{x}_{max} = 2\sqrt{10}$

SIMPLE HARMONIC MOTION (SMH)

3 A particle is moving in a straight line. If x metres is its displacement at time t seconds and $\left(\frac{dx}{dt}\right)^2 = 5(4 - x^2)$, find the acceleration in terms of x only. Show that the motion is simple harmonic and find its period and amplitude.

$[\dot{x}(t)]^2 = 5[4 - x^2]$ we differentiate both sides

$\frac{d}{dt}[\dot{x}(t)]^2 = \frac{d}{dt}5[4 - x^2]$ so $2\frac{d}{dt}[\dot{x}(t)] \times \dot{x}(t) = -\frac{d}{dt}(5x^2)$

$\Leftrightarrow 2\ddot{x}(t) \times \dot{x}(t) = -5 \cdot 2\frac{dx(t)}{dt} \times x(t) = -10\,\dot{x}(t) \times x(t)$

so $\ddot{x}(t) = -5\,x(t)$

The equation is of the form $\ddot{x} = -kx$, $\therefore$ the particle motion is simple harmonic.

$x(t) = a\cos(nt + \alpha)$

$\ddot{x}(t) = -an^2\cos(nt + \alpha)$ so $n^2 = 5$, $n = \sqrt{5}$

and the period is $\frac{2\pi}{\sqrt{5}}$

$\left(\frac{dx}{dt}\right)^2 = [an\sin(nt+\alpha)]^2 = a^2n^2\sin^2(nt+\alpha) = 5a^2\sin^2(nt+\alpha)$

$5(4 - x^2) = 5\left(4 - a^2\cos^2(nt+\alpha)\right)$

So $a^2\sin^2(nt+\alpha) = 4 - a^2\cos^2(nt+\alpha)$

so we must have $a^2 = 4$, i.e. $a = 2$

SIMPLE HARMONIC MOTION (SMH)

8 A particle is moving along the x-axis in simple harmonic motion centred at the origin.
When $x = 2$, the velocity of the particle is 5.
When $x = 5$, the velocity of the particle is 4. Find:

(a) the amplitude of the motion **(b)** the period of the motion.

$x(t) = a\cos(nt+\alpha)$ $\qquad$ $\dot{x}(t) = -an\sin(nt+\alpha)$

so $a^2\cos^2(nt+\alpha) = [x(t)]^2$ and $a^2\sin^2(nt+\alpha) = \dfrac{[\dot{x}(t)]^2}{n^2}$

So $a^2 = [x(t)]^2 + \dfrac{[\dot{x}(t)]^2}{n^2}$

At $x = 2$, $\dot{x} = 5$, i.e $a^2 = 2^2 + \dfrac{5^2}{n^2} = 4 + \dfrac{25}{n^2}$

At $x = 5$, $\dot{x} = 4$, i.e. $a^2 = 5^2 + \dfrac{4^2}{n^2} = 5^2 + \dfrac{16}{n^2}$

So by elimination $4 + \dfrac{25}{n^2} = 25 + \dfrac{16}{n^2}$

$\Longleftrightarrow \dfrac{25-16}{n^2} = 25 - 4 = 21 \Longleftrightarrow \dfrac{9}{n^2} = 21$

So $n^2 = \dfrac{9}{21}$ $\qquad n = \dfrac{3}{\sqrt{21}} = \dfrac{3\sqrt{21}}{21} = \dfrac{\sqrt{21}}{7}$ $\qquad$ Period is $\dfrac{2\pi}{\left(\frac{\sqrt{21}}{7}\right)} = \dfrac{14\pi}{\sqrt{21}} = \dfrac{2\pi\sqrt{21}}{3}$

and $a^2 = 4 + \dfrac{25}{\left(\frac{9}{21}\right)} = 4 + \dfrac{21 \times 25}{9} = \dfrac{187}{3}$

so $a = \sqrt{\dfrac{187}{3}}$ amplitude

SIMPLE HARMONIC MOTION (SMH)

9 A particle moves in a straight line. At time t seconds, its displacement x cm from a fixed point O in the line is given by $x = 5\cos\left(\frac{\pi}{2}t - \frac{\pi}{3}\right)$. Express the acceleration in terms of x only and hence show that the motion is simple harmonic. Find:

(a) the period **(b)** the amplitude **(c)** the speed when $x = -2.5$ **(d)** the acceleration when $x = -2.5$.

$$\dot{x}(t) = 5 \times \frac{\pi}{2} \times \left[-\sin\left(\frac{\pi}{2}t - \frac{\pi}{3}\right)\right]$$

$$\ddot{x}(t) = -\frac{5\pi}{2} \times \frac{\pi}{2} \cos\left(\frac{\pi}{2}t - \frac{\pi}{3}\right) = -\frac{\pi^2}{4} x(t)$$

The motion is of the form $\ddot{x} = -kx$, $\therefore$ it's a Simple Harmonic Motion.

a) Period is $\frac{2\pi}{\pi/2} = 2\pi \div \frac{\pi}{2} = 2\pi \times \frac{2}{\pi} = 4$ s

b) Amplitude is 5 cm

c) when $x = -2.5$ $\quad -2.5 = 5\cos\left(\frac{\pi}{2}t - \frac{\pi}{3}\right)$

so $\cos\left(\frac{\pi}{2}t - \frac{\pi}{3}\right) = -\frac{1}{2}$ so $\frac{\pi}{2}t - \frac{\pi}{3} = \frac{2\pi}{3}$ so $\frac{\pi}{2}t = \pi$ $t = 2$ s

$$\dot{x}(2) = -\frac{5\pi}{2}\sin\left(\frac{\pi}{2} \times 2 - \frac{\pi}{3}\right) = -\frac{5\pi}{2}\sin\left(\frac{2\pi}{3}\right) = -\frac{5\pi}{2} \times \frac{\sqrt{3}}{2} = \frac{-5\pi\sqrt{3}}{4}$$

d) $\dot{x}(2) = -\frac{\pi^2}{4} \times (-2.5) = \frac{2.5\pi^2}{4} = \frac{5\pi^2}{8}$ cm s^{-2}

SIMPLE HARMONIC MOTION (SMH)

14 Solve the differential equation $\frac{d^2x}{dt^2}+16x=0$ subject to the conditions $x=3$ and $\frac{dx}{dt}=16$ when $t=0$. Find the maximum displacement and the maximum speed if x metres is the displacement of the particle moving in a straight line at time t seconds.

$\frac{d^2x}{dt^2}+16x=0 \iff \ddot{x}(t)=-16x$ which is the differential equation for a SHM, and $\therefore$ the solution is $x(t)=a\cos(4t+\alpha)$.

as $n^2=16$

At $t=0$, $x=3$ so $a\cos\alpha=3$

and $\frac{dx}{dt}=16$ so $-a\times 4\sin\alpha=16$

so $a\sin\alpha=-4$

$\therefore$ $a^2=9+16=25$ so $a=5$

and $\cos\alpha=3/5$ $\sin\alpha=-4/5$

so $x(t)=5\cos(4t+\alpha)$

$x(t)=5\left[\cos 4t\cos\alpha-\sin 4t\sin\alpha\right]$

$x(t)=5\left[\frac{3}{5}\cos 4t+\frac{4}{5}\sin 4t\right]$

$x(t)=3\cos 4t+4\sin 4t$

Maximum displacement is $5\,m$

Maximum speed is $4\times 5=20\ m\,s^{-1}$

SIMPLE HARMONIC MOTION (SHM)

22 A floating buoy oscillates up and down with the waves, rising and falling 2 metres about its mean position. Find its greatest velocity and acceleration if the period of the motion is 3 seconds.

$x(t) = 2\sin(nt+\alpha)$

$\dot{x}(t) = 2n\cos(nt+\alpha)$

Taking that at $t=0$, $x=0$, so $\alpha = 0$

$x(t) = 2\sin(nt)$

$\dot{x}(t) = 2n\cos(nt)$

Period is $3s$, so as $T = \frac{2\pi}{n}$, then $n = \frac{2\pi}{T} = \frac{2\pi}{3}$

$\dot{x}(t) = 2 \times \frac{2\pi}{3}\cos(nt) = \frac{4\pi}{3}\cos\left(\frac{2\pi t}{3}\right)$

So the greatest velocity is $\frac{4\pi}{3}$ ms^{-1}

$\ddot{x}(t) = \frac{4\pi}{3} \times \frac{2\pi}{3}\left[-\sin\left(\frac{2\pi}{3}t\right)\right] = -\frac{8\pi^2}{9}\sin\left(\frac{2\pi}{3}t\right)$

So the greatest acceleration is $\frac{8\pi^2}{9}$ ms^{-1}

SIMPLE HARMONIC MOTION (SHM)

29 A point moves with SHM in such a way that its speed is 8 and 6 m s^{-1} respectively at distances 3 and 4 m from the mean position. Calculate the period of the motion and the magnitude of the greatest acceleration.

$$x(t) = a\cos(nt+\alpha) \iff \cos(nt+\alpha) = \frac{x}{a}$$

$$\dot{x}(t) = -an\sin(nt+\alpha) \iff \sin(nt+\alpha) = -\frac{V}{an}$$

So $\left[\cos(nt+\alpha)\right]^2 = \left(\frac{x}{a}\right)^2$ and $\left[\sin(nt+\alpha)\right]^2 = \left[\frac{-V}{an}\right]^2 = \left(\frac{V}{an}\right)^2$

Hence $1 = \left(\frac{x}{a}\right)^2 + \left(\frac{V}{an}\right)^2$

When $x = 3$, $v = 8$, i.e. $1 = \left(\frac{3}{a}\right)^2 + \left(\frac{8}{an}\right)^2$ Equation ①

When $x = 4$, $v = 6$, i.e. $1 = \left(\frac{4}{a}\right)^2 + \left(\frac{6}{an}\right)^2$ Equation ②

① $\iff a^2 = 9 + \frac{64}{n^2}$ ② $\iff a^2 = 16 + \frac{36}{n^2}$

So $9 + \frac{64}{n^2} = 16 + \frac{36}{n^2}$

$\iff \frac{64-36}{n^2} = 16 - 9 = 7$ so $\frac{28}{n^2} = 7$ so $n^2 = 4$ $\boxed{n = 2}$

Period is $T = \frac{2\pi}{n} = \frac{2\pi}{2} = \pi$ seconds.

$a^2 = 9 + \frac{64}{n^2} = 9 + \frac{64}{2^2} = 25$ so $a = 5$ m

Greatest acceleration is $an^2 = 5 \times 2^2 = 20$ m s^{-2}

SIMPLE HARMONIC MOTION (SMH)

37 A particle is moving in a straight line under simple harmonic motion. It has a displacement of x metres from a point O, on the line, at time t seconds given by $x = 1 + 2\cos\left(2t - \frac{\pi}{4}\right)$.

(a) Show that $\ddot{x} = -4(x-1)$.

(b) Find the centre of the motion and the time taken for the particle to first reach maximum speed.

(c) Find the amplitude of the motion and when the particle is first at rest.

$x(t) = 1 + 2\cos\left(2t - \frac{\pi}{4}\right)$

$\dot{x}(t) = -2 \times 2 \sin\left(2t - \frac{\pi}{4}\right)$ $\quad\quad$ $\ddot{x}(t) = -8\cos\left(2t - \frac{\pi}{4}\right)$

a) $-4(x-1) = -4\left[1 + 2\cos\left(2t - \frac{\pi}{4}\right) - 1\right] = -8\cos\left(2t - \frac{\pi}{4}\right) = \ddot{x}(t)$

b) The centre of motion is $x = 1$

The speed is maximum for $\dot{x}(t) = -4$, i.e. $\sin\left(2t - \frac{\pi}{4}\right) = +1$

$\Leftrightarrow \sin\left(2t - \frac{\pi}{4}\right) = \sin\frac{\pi}{2}$ $\quad$ so $2t - \frac{\pi}{4} = \frac{\pi}{2}$

$2t = \frac{3\pi}{4}$ $\quad\quad$ $t = \frac{3\pi}{8}$ s.

c) Amplitude is 2 m

$\dot{x}(t) = 0$ when $\sin\left(2t - \frac{\pi}{4}\right) = 0$

i.e. $2t = \frac{\pi}{4}$ $\quad\quad$ $t = \pi/8$ seconds.

38 The tide can be modelled using simple harmonic motion. At a particular location, the depth at high tide is 5 metres and the depth at low tide is 1 metre. At this location, the tide completes two full periods every 25 hours. Let x represent the depth in metres and t be the time in hours after the first low tide of the day.

(a) If this depth of this tide can be modelled by the function $x = a\cos nt + c$, find the values of a, n and c. The first low tide today is at 2 a.m.

(b) At what time is the first high tide today?

(c) At what time this evening is the depth of water increasing at the fastest rate?

a) Amplitude is 2 m Period is $\frac{25}{2} = \frac{2\pi}{n}$ so $n = \frac{4\pi}{25}$

Halfway between 1 and 5 is 3, so $c = 3$

Hence $x(t) = \pm 2\cos\left(\frac{4\pi t}{25}\right) + 3$ which one?

At $t=0$, $x(0) = 1$, so $1 = \pm 2 + 3$ true with -2
(i.e at 2am)
so $x(t) = 3 - 2\cos\left(\frac{4\pi t}{25}\right)$

b) The period is 12.5 hours.
The low tide is at 2 am, so the first high tide is at
$2 + 6.25 = 8.25$
or 8.15 am

c) The tide is low again at $2 + 12.5 = 14.5$
so 2.30 pm.

The tide is high again at $14.5 + 6.25 = 20.75$ or 8.45 pm

The depth of the water is increasing at its fastest rate halfway between the low and the high tide, i.e.

at $\frac{20.75 + 14.5}{2} = 17.625$ which is 5.37 pm