SOLVING QUADRATIC EQUATIONS WITH COMPLEX COEFFICIENTS

To solve the quadratic equation $a\,x^2 + b\,x + c = 0$, where a, b and c are real coefficients, we use the quadratic formula $x = \frac{-b \pm \sqrt{\Delta}}{2a}$. When a, b and c are real coefficients, the roots will be either real or complex. If they are complex, they will occur as conjugate pairs z and $\bar{z}$ (because of the conjugate root theorem for polynomial functions with real coefficients).

If a, b and c are complex, this formula does not generate a simple solution, so we need to use another method to find the roots of the quadratic expression; the method to be used in that case is by completing the square, as follows:

Solve the equation by completing the square.

$$ax^2 + bx + c = 0 \qquad [1]$$

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a} - c$$

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a}$$

$$\left[2a\left(x + \frac{b}{2a}\right)\right]^2 = b^2 - 4ac \qquad [2]$$

If the original equation, [1], has complex roots, then equation [2] says that you have a complex number whose square is equal to $b^2 - 4ac$.

Let this complex number be $\alpha + i\beta$.

Thus equation [2] becomes $(\alpha + i\beta)^2 = A + iB$ [3] and you have to obtain α and β as real numbers.

$$(\alpha + i\beta)^2 = A + iB$$

$$\alpha^2 + 2i\alpha\beta - \beta^2 = A + iB$$

$$(\alpha^2 - \beta^2) + 2\alpha\beta i = A + Bi$$

If two complex numbers are equal, then their real parts and their imaginary parts are equal.

$$\therefore\ \alpha^2 - \beta^2 = A \qquad [i]$$

$$2\alpha\beta = B \qquad [ii]$$

Solve [i] and [ii] for α and β.

Now $\left(\alpha^2 + \beta^2\right)^2 = \alpha^4 + 2\alpha^2\beta^2 + \beta^4$

$$= \alpha^4 - 2\alpha^2\beta^2 + \beta^4 + 4\alpha^2\beta^2$$

$$= \left(\alpha^2 - \beta^2\right)^2 + (2\alpha\beta)^2$$

$$= A^2 + B^2$$

Since $\alpha^2 + \beta^2 > 0$ then $\alpha^2 + \beta^2 = \sqrt{A^2 + B^2}$ [iii]

[i] + [iii] $\quad 2\alpha^2 = A + \sqrt{A^2 + B^2}$

$$\alpha^2 = \frac{A + \sqrt{A^2 + B^2}}{2}$$

$$\therefore \qquad \alpha = \pm\sqrt{\frac{A + \sqrt{A^2 + B^2}}{2}}$$

[ii] gives: $\beta = \frac{B}{2\alpha}$

If $\alpha = \sqrt{\frac{A + \sqrt{A^2 + B^2}}{2}}$, then $\beta = \frac{B}{2}\sqrt{\frac{2}{A + \sqrt{A^2 + B^2}}}$

and if: $\alpha = -\sqrt{\frac{A + \sqrt{A^2 + B^2}}{2}}$, then $\beta = -\frac{B}{2}\sqrt{\frac{2}{A + \sqrt{A^2 + B^2}}}$

Example 25(a): Solve $x^2 + 2x + i = 0$

Solution

(a) $x^2 + 2x + i = 0$

Complete the square: $x^2 + 2x + 1 = 1 - i$

$(x+1)^2 = 1 - i$

Let $x + 1 = \alpha + \beta i$

$\therefore (\alpha + \beta i)^2 = 1 - i$

$\alpha^2 - \beta^2 + 2\alpha\beta i = 1 - i$

$\therefore \alpha^2 - \beta^2 = 1$

$2\alpha\beta = -1$

Now $\left(\alpha^2 + \beta^2\right)^2 = \left(\alpha^2 - \beta^2\right)^2 + (2\alpha\beta)^2$

$= 1^2 + (-1)^2$

$= 2$

$\therefore \alpha^2 + \beta^2 = \sqrt{2}$

and $\alpha^2 - \beta^2 = 1$

$\therefore \quad 2\alpha^2 = \sqrt{2} + 1$

$\alpha^2 = \frac{\sqrt{2}+1}{2}$

$\alpha = \pm\sqrt{\frac{\sqrt{2}+1}{2}}$

$\beta = \frac{-1}{2\alpha}$

When $\alpha = \sqrt{\frac{\sqrt{2}+1}{2}}$ then $\beta = \frac{-1}{2} \times \frac{\sqrt{2}}{\sqrt{\sqrt{2}+1}} = \frac{-1}{\sqrt{2+\sqrt{2}}} = -\frac{\sqrt{2-\sqrt{2}}}{\sqrt{4-2}} = -\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2}} = -\sqrt{\sqrt{2}-1}$

and when $\alpha = -\sqrt{\frac{\sqrt{2}+1}{2}}$ then $\beta = \sqrt{\sqrt{2}-1}$

Hence $x + 1 = \sqrt{\frac{\sqrt{2}+1}{2}} - i\sqrt{\sqrt{2}-1}$ and $x + 1 = -\sqrt{\frac{\sqrt{2}+1}{2}} + i\sqrt{\sqrt{2}-1}$

$x = \left(\sqrt{\frac{\sqrt{2}+1}{2}} - 1\right) - i\sqrt{\sqrt{2}-1}$ $\quad x = -\left(\sqrt{\frac{\sqrt{2}+1}{2}} + 1\right) + i\sqrt{\sqrt{2}-1}$

Example 25(b): Solve $x^2 + 2(2 - i)x + 6 = 0$

(b) $x^2 + 2(2 - i)x + 6 = 0$

Complete the square: $x^2 + 2(2 - i)x + (2 - i)^2 = (2 - i)^2 - 6$

$$(x + 2 - i)^2 = 4 - 4i - 1 - 6$$

$$(x + 2 - i)^2 = -3 - 4i$$

Let $x + 2 - i = \alpha + \beta i$

$\therefore\ (\alpha + \beta i)^2 = -3 - 4i$

$\alpha^2 - \beta^2 + 2\alpha\beta i = -3 - 4i$

$\therefore\ \alpha^2 - \beta^2 = -3$

$2\alpha\beta = -4$

Now $\left(\alpha^2 + \beta^2\right)^2 = \left(\alpha^2 - \beta^2\right)^2 + \left(2\alpha\beta\right)^2$

$= (-3)^2 + (-4)^2$

$= 25$

$\therefore\ \alpha^2 + \beta^2 = 5$

and $\alpha^2 - \beta^2 = -3$

$2\alpha^2 = 2$

$\alpha^2 = 1$

$\alpha = 1, \beta = -2$

$\alpha = -1, \beta = 2$

The roots of the equation are $x + 2 - i = 1 - 2i$ and $x + 2 - i = -1 + 2i$

i.e. $x = -1 + i$ and $x = -3 + 3i$

Example 25(c): Solve $i\,x^2 + 3\,x - 4 = 0$

(c) $ix^2 + 3x - 4 = 0$

Multiply by $-i$: $x^2 - 3ix + 4i = 0$

Complete the square: $x^2 - 3ix + \left(\frac{3i}{2}\right)^2 = \left(\frac{3i}{2}\right)^2 - 4i$

$$\left(x - \frac{3i}{2}\right)^2 = -\frac{9}{4} - 4i$$

Let $x - \frac{3i}{2} = \alpha + \beta i$

$$\therefore \quad (\alpha + \beta i)^2 = -\frac{9}{4} - 4i$$

$$\alpha^2 - \beta^2 + 2\alpha\beta i = -\frac{9}{4} - 4i$$

$$\therefore \quad \alpha^2 - \beta^2 = -\frac{9}{4}$$

$$2\alpha\beta = -4$$

$$\left(\alpha^2 + \beta^2\right)^2 = \left(\alpha^2 - \beta^2\right)^2 + (2\alpha\beta)^2$$

$$= \left(-\frac{9}{4}\right)^2 + (-4)^2$$

$$= \frac{337}{16}$$

Now $\therefore \quad \alpha^2 + \beta^2 = \frac{\sqrt{337}}{4}$

$$\alpha^2 - \beta^2 = -\frac{9}{4}$$

$$2\alpha^2 = \frac{\sqrt{337} - 9}{4}$$

$$\alpha^2 = \frac{\sqrt{337} - 9}{8}$$

$$\alpha = \frac{\sqrt{\sqrt{337} - 9}}{2\sqrt{2}} \text{ and } \beta = \frac{-2}{\alpha} = \frac{-4\sqrt{2}}{\sqrt{\sqrt{337} - 9}} = \frac{-4\sqrt{2} \times \sqrt{\sqrt{337} + 9}}{\sqrt{337 - 81}} = \frac{-4\sqrt{2} \times \sqrt{\sqrt{337} + 9}}{\sqrt{256}} = \frac{-\sqrt{2} \times \sqrt{\sqrt{337} + 9}}{4}$$

Rationalise so that when $\alpha = \frac{\sqrt{2\left(\sqrt{337} - 9\right)}}{4}$, $\beta = \frac{-\sqrt{2\left(\sqrt{337} - 9\right)}}{4}$

and when $\alpha = \frac{-\sqrt{2\left(\sqrt{337} - 9\right)}}{4}$, $\beta = \frac{\sqrt{2\left(\sqrt{337} - 9\right)}}{4}$

Thus $x - \frac{3i}{2} = \frac{\sqrt{2\left(\sqrt{337} - 9\right)}}{4} + \frac{-\sqrt{2\left(\sqrt{337} - 9\right)}}{4}i$ and $x - \frac{3i}{2} = \frac{-\sqrt{2\left(\sqrt{337} - 9\right)}}{4} + \frac{\sqrt{2\left(\sqrt{337} - 9\right)}}{4}i$

$$x = \frac{\sqrt{2\left(\sqrt{337} - 9\right)}}{4} + \frac{6 - \sqrt{2\left(\sqrt{337} - 9\right)}}{4}i \qquad x = \frac{-\sqrt{2\left(\sqrt{337} - 9\right)}}{4} + \frac{6 + \sqrt{2\left(\sqrt{337} - 9\right)}}{4}i$$

Square root of a complex number

The example below demonstrates the method for finding the square root of a complex number.

Example 26

Find $\sqrt{4+3i}$.

Solution

Let $z = x + iy$, where x, y are real, such that $z^2 = 4 + 3i$

Hence $(x+iy)^2 = 4 + 3i$

Expand: $x^2 - y^2 + 2xyi = 4 + 3i$

Equating the real and imaginary parts of the LHS and RHS gives:

$x^2 - y^2 = 4$ [1] and

$2xy = 3$ [2]

Now $(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$: $(x^2+y^2)^2 = 4^2 + 3^2 = 25$

$x^2 + y^2 = 5$ [3]

$x^2 - y^2 = 4$ [1]

[3] + [1]: $2x^2 = 9$

$$x = \pm\frac{3}{\sqrt{2}} = \pm\frac{3\sqrt{2}}{2}$$

From [2]: $y = \dfrac{3}{2x}$

$$y = \frac{3}{2}\times\left(\pm\frac{\sqrt{2}}{3}\right) = \pm\frac{\sqrt{2}}{2}$$

Hence $\sqrt{4+3i} = \dfrac{3\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}i$ or $-\dfrac{3\sqrt{2}}{2} - \dfrac{\sqrt{2}}{2}i$