OTHER INDUCTION QUESTIONS

Use mathematical induction to prove the following results.

1 $n^2 - 11n + 30 \geq 0$ for all integers $n \geq 6$.

2 $n^2 > -5n + 14$ for all integers $n > 2$.

① Step 1 : for $n=6$ $n^2 - 11n + 30 = 36 - 11 \times 6 + 30 = 0$ so true.

Step 2 : Assume it's true for $n = k$.

So for $n = k+1$ $(k+1)^2 - 11(k+1) + 30 = k^2 + 2k + 1 - 11k - 11 + 30$

$$= \underbrace{k^2 - 11k + 30}_{\geq 0 \text{ by assumption}} + \underbrace{2k - 10}_{\geq 0 \text{ as } k \geq 6.}$$

So it's true for $k+1$ if it's true for k.

Step 3 : it's true for $n = 6$

it's true for $k+1$ if it's true for k. $\therefore$ it's true for any $k \geq 6$

② $n^2 > -5n + 14 \iff n^2 + 5n - 14 > 0$

Step 1 for $n = 3$ $3^2 + 5 \times 3 - 14 = 9 + 15 - 14 > 0$ indeed.

Step 2 Assume it's true for $n = k$

In that case $(k+1)^2 + 5(k+1) - 14 = k^2 + 2k + 1 + 5k + 5 - 14$

$$= \underbrace{k^2 + 5k - 14}_{>0 \text{ by assumption}} + \underbrace{2k + 6}_{>0 \text{ as } k > 0}$$

So it's true for $(k+1)$ if it's true for k

Step 3 it's true for $n = 3$

it's true for $(k+1)$ if it's true for k.

$\therefore$ by induction, it's true for any $n \geq 3$

OTHER INDUCTION QUESTIONS

8 **(a)** Prove that $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$ for any positive integer n by:

(i) first proving S(1) that $\frac{d}{dx}(x) = 1$

(ii) then writing $x^{n+1} = x \times x^n$ and using the product rule to prove that $S(k+1)$ is true.

(b) Summarise your results to give the proof of the result by induction.

a) i) $\frac{d}{dx}(x) = 1$ easily proven using first principle of differentiation

ii) $x^{n+1} = x \times x^n$ so $\frac{d}{dx}\left(x^{n+1}\right) = \frac{d}{dx}\left(x \times x^n\right) = \frac{d(x)}{dx} x^n + x\frac{dx^n}{dx}$

So, assuming it's true for k, then

$\frac{dx^{k+1}}{dx} = 1 \times x^k + x \times \underbrace{k\,x^{k-1}}_{\text{by assumption}} = x^k + k\,x^k = x^k(k+1)$

So it's also true for $(k+1)$

b) it's true for $n=1$

it's true for $(k+1)$ if it's true for k.

$\therefore$ by induction, it's true for any $n \geqslant 1$

OTHER INDUCTION QUESTIONS

14 Prove that $\frac{d^n}{dx^n}(x^n) = n!$ for integral n, $n \geq 0$.

Step 1 $\quad n = 1 \quad \frac{d}{dx} x = 1! = 1$ indeed, so true for $n = 1$

Step 2 Assume it's true for $n = k$, i.e. $\frac{d^k}{dx^k}(x^k) = k!$

$$\frac{d^{k+1}}{dx^{k+1}}(x^{k+1}) = \frac{d^k}{dx^k}\left[\frac{d}{dx}(x^{k+1})\right] = \frac{d^k}{dx^k}\left[\frac{d}{dx}(x \times x^k)\right]$$

$$\text{——} = \frac{d^k}{dx^k}\left[x \frac{d}{dx} x^k + 1 \times x^k\right]$$

$$\text{——} = \frac{d^k}{dx^k}\left[x \times k x^{k-1} + x^k\right]$$

$$\text{——} = \frac{d^k}{dx^k}\left[k x^k\right] + \frac{d^k}{dx^k}(x^k)$$

$$\text{——} = k \frac{d^k}{dx^k}(x^k) + k!$$

by assumption.

$$\text{——} = k \times k! + k!$$

$$\text{——} = k!(k+1) = (k+1)!$$

So it's true for $(k+1)$ if it's true for k

Step 3 it's true for $k = 1$

it's true for $(k+1)$ if it's true for k.

$\therefore$ by induction, it's true for any $n \geq 1$

OTHER INDUCTION QUESTIONS

15 The binomial theorem states that if n is an integer, $n \geq 1$, then $(x+a)^n = \sum_{r=0}^{n} {}^nC_r x^r a^{n-r}$. Use mathematical induction to prove this result.

$${}^nC_r = \frac{n!}{(n-r)!\,r!}$$

Step 1: for $n=1$ $(x+a)^1 = x+a$

whereas $\sum_{r=0}^{1} {}^1C_r x^r a^{1-r} = {}^1C_0 x^0 a + {}^1C_1 x^1 a^0$
$= a + x$ so true for $n=1$

Step 2. we assume it's true for $n=k$

i.e $(x+a)^k = \sum_{r=0}^{k} {}^kC_r x^r a^{k-r}$

Now for $n=k+1$.

$$(x+a)^{k+1} = (x+a)(x+a)^k = x\sum_{r=0}^{k} {}^kC_r x^r a^{k-r} + a\sum_{r=0}^{k} {}^kC_r x^r a^{k-r}$$

$$= \sum_{r=0}^{k} {}^kC_r x^{r+1} a^{k-r} + \sum_{r=0}^{k} {}^kC_r x^r a^{k-r+1}$$

$$= \sum_{r=1}^{k+1} {}^kC_{r-1} x^r a^{k-r+1} + \sum_{r=0}^{k} {}^kC_r x^r a^{k-r+1}$$

$$= {}^kC_0 x^0 a^{k+1} + \sum_{r=1}^{k} \left[{}^kC_{r-1} + {}^kC_r\right] x^r a^{k-r+1} + {}^{k+1}C_{k+1-1}\, x^{k+1} a^{k+1-(k+1)}$$

$$= a^{k+1} + \sum_{r=1}^{k} \left[{}^kC_{r-1} + {}^kC_r\right] x^r a^{k-r+1} + x^{k+1}$$

But $${}^kC_{r-1} + {}^kC_r = \frac{k!}{(r-1)!\,(k-(r-1))!} + \frac{k!}{r!\,(k-r)!}$$

$$= \frac{k!}{(r-1)!\,[(k+1)-r]!} + \frac{k!}{r!\,(k-r)!}$$

$$= \frac{k!\,r + k!\,[(k+1)-r]}{r!\,[(k+1)-r]!} = \frac{k!\,[r+(k+1)-r]}{r!\,[(k+1)-r]!}$$

So ${}^kC_{r-1} + {}^kC_r = {}^{k+1}C_r$

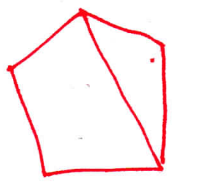
$$= a^{k+1} + \sum_{r=1}^{k} {}^{k+1}C_r x^r a^{k+1-r} + x^{k+1}$$

$$= \sum_{r=0}^{k+1} {}^{k+1}C_r x^r a^{k+1-r} \quad \therefore \text{ true for } k+1$$

Step 3: true for $n=1$. True for $k+1$ if true for k
$\therefore$ by induction, true for any $n \geq 1$

OTHER INDUCTION QUESTIONS

17 Prove that the number of diagonals of a convex polygon with n vertices is $\frac{n(n-3)}{2}$ for $n \geq 4$.



Step 1 True for $n = 4$ as 2 diagonals and $\frac{4 \times (4-3)}{2} = 2$ indeed.

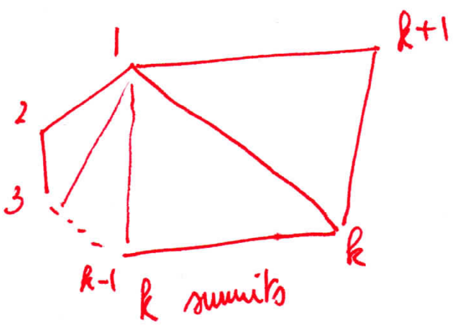
Step 2 . Assume it's true for $n = k$
i.e a convex polygon with k vertices has $\frac{k(k-3)}{2}$ diagonals.
Now a convex polygon with $(k+1)$ vertices:
each vertice would have one additional diagonal, so $(k-1)$ more diagonals,
So the total number of diagonals would be $\frac{k(k-3)}{2} + k - 1$.

$$\frac{k(k-3)}{2} + k - 1 = \frac{k(k-3)+2k-2}{2} = \frac{k^2-k-2}{2} = \frac{(k+1)(k-2)}{2}$$

So it's equal to $\frac{(k+1)[(k+1)-3]}{2}$

So it's true for $k+1$

Step 3 it's true for $n = 4$
it's true for $k+1$ if it's true for k
$\therefore$ by induction, it's true for any $n \geq 4$



Each of the k summits has an extra diagonal (so k in total) except summit k which doesn't (already counted for summit 1.
So only $(k-1)$ extra diagonals

OTHER INDUCTION QUESTIONS

23 (a) Write the binomial expansion of $(k+1)^p$ where p is a positive integer.

(b) If p is a prime number, identify which of the terms in the expansion do not have a factor of p.

(c) Prove by induction on n that if n is a positive integer and p is a prime number, then $n^p - n$ is a multiple of p.

a) $(k+1)^p = \sum_{r=0}^{p} {}^pC_r\, k^r\, 1^{p-r} = \sum_{r=0}^{p} {}^pC_r\, k^r = \sum_{r=0}^{p} \frac{p!}{(p-r)!\,r!} k^r$

b) The terms in the expansion which do not have a factor of p are the constant term (1) and k^p

c) For $n=1$ $\quad n^p - n = 1 - 1 = 0$ is a multiple of p.

$n=2 \quad n^p - n = 2^p - 2 = 2(2^{p}-1)$

Step 2: $k^p - k$ is a multiple of p. so $k^p - k = qp$. For some integer q

Now for $k+1$ $\quad (k+1)^p - (k+1) = \underbrace{\sum_{r=0}^{p} \frac{p!}{(p-r)!\,r!} k^r}_{} - (k+1)$

in this expansion, the only terms which do not have a factor of p are the term k^p and 1; all others are factors of p. So:

$$(k+1)^p - (k+1) = \underline{\text{Factors of } p} + k^p + 1 - k - 1$$

$$\underline{\quad\quad} = \underline{\text{Factors of } p} + k^p - k + 1 - 1$$

$$\underline{\quad\quad} = \underbrace{\text{Factors of } p + \underbrace{qp}_{\text{by assumption}}}_{\text{So this term is a Multiple of } p}$$

$\therefore (k+1)^p - (k+1)$ is a multiple of p.

Step 3: it's true for $n=1$

it's true for $(k+1)$ if it's true for k

$\therefore$ by induction, it's true for any n positive integer, i.e. if n a positive integer and p a prime number, then $n^p - n$ is a multiple of p.