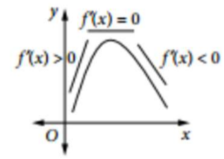
THE FIRST DERIVATIVE AND TURNING POINTS

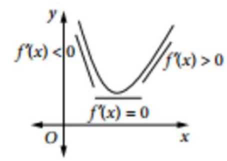
In functions you have seen so far, a stationary point is usually the point where the function changes from increasing to decreasing (or from decreasing to increasing), so that the stationary point is the highest (or lowest) point in the neighbourhood. (Here 'neighbourhood' means 'near the point on the graph'.) However, sometimes the function does not change the sign of its gradient at the stationary point. This situation will be considered later.

When a function changes from increasing to decreasing at a stationary point, the sign of $f'(x)$ changes from positive to negative.



When a function changes from decreasing to increasing at a stationary point, the sign of $f'(x)$ changes from negative to positive.

These points are called **turning points**. If the turning point is higher than the other points in its neighbourhood, it is called a **local maximum** turning point. If the point is lower than the other points in its neighbourhood, it is called a **local minimum** turning point.



A turning point of $f(x)$ is a point where the curve $y = f(x)$ is locally a maximum or a minimum.

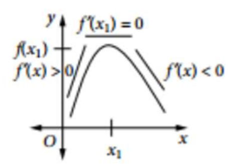
For a differentiable function $f(x)$, all turning points are stationary points.

(However, note that not all stationary points are turning points.)

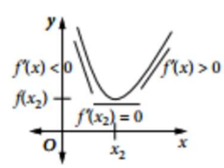
First derivative test for local maxima and minima

A turning point of the differentiable function $f(x)$ may occur when $f'(x) = 0$. The type of turning point will depend on the change in sign of $f'(x)$ as x passes through the abscissa (x-coordinate) of the stationary point, so you need to find the sign of $f'(x)$ on either side of the stationary point.

If $f'(x) = 0$ when $x = x_1$, $f'(x) > 0$ when $x < x_1$, and $f'(x) < 0$ when $x > x_1$, then the sign of $f'(x)$ changes from positive to negative as x passes through x_1. The point $(x_1, f(x_1))$ must be a **local maximum turning point**.



If $f'(x) = 0$ when $x = x_2$, $f'(x) < 0$ when $x < x_2$, and $f'(x) > 0$ when $x > x_2$, then the sign of $f'(x)$ changes from negative to positive as x passes through x_2. The point $(x_2, f(x_2))$ is a **local minimum turning point**.



Thus there are two conditions required to find turning points of the function $y = f(x)$:

1. $f'(x) = 0$ at $x = x_1$, and
2. $f'(x)$ changes sign as x passes through x_1.

The way that $f'(x)$ changes will tell you whether the turning point is a maximum or minimum turning point.

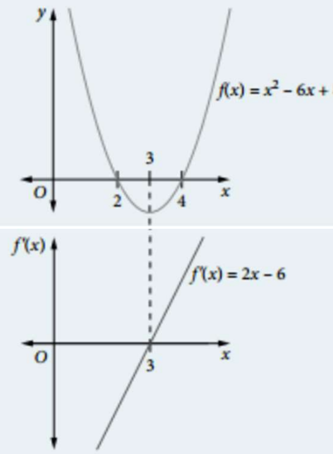
Example 4

The diagrams show the graphs of $f(x) = x^2 - 6x + 8$ and $f'(x) = 2x - 6$ drawn with a common x scale.

(a) Find the coordinates of any stationary points on $f(x)$.

(b) Use the graph of $f'(x)$ to determine the nature of the turning points (i.e. maximum or minimum).

(c) What is the least value of $f(x)$?



Solution

(a) The graph of $f'(x)$ gives that $f'(x) = 0$ when $x = 3$.

$f(3) = 9 - 18 + 8 = -1$

The coordinates of the stationary point are $(3, -1)$.

(b) When $x < 3$, $f'(x) < 0$; when $x > 3$, $f'(x) > 0$.

$f'(x)$ changes from negative to positive as x passes through 3.

$\therefore$ The stationary point is a relative minimum turning point (local minimum).

(c) The least value of $f(x)$ is -1.

Example 5

A function is given by $f(x) = x^3 - 12x + 16$.

(a) Find $f'(x)$.

(b) Find the coordinates of any stationary points.

(c) Determine the nature of the stationary points.

(d) Sketch $y = f(x)$.

Solution

(a) $f'(x) = 3x^2 - 12$

(b) For stationary points, $f'(x) = 0$, so:

$$3(x^2 - 4) = 0$$
$$(x + 2)(x - 2) = 0$$
$$x = -2, 2$$

$f(-2) = -8 + 24 + 16 = 32$ $\quad f(2) = 8 - 24 + 16 = 0$

Stationary points are $(-2, 32)$ and $(2, 0)$.

(c) Consider the stationary point $(-2, 32)$.

At $x = -3$, $f'(-3) = 27 - 12 = 15 > 0$

At $x = -1$, $f'(-1) = 3 - 12 = -9 < 0$

$f'(x)$ changes from +ve to −ve, so $f(x)$ has a maximum turning point at $(-2, 32)$.

Consider the stationary point $(2, 0)$.

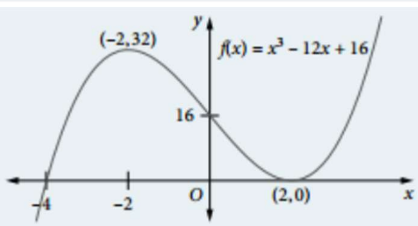
At $x = 1$, $f'(1) = 3 - 12 = -9 < 0$

At $x = 3$, $f'(3) = 27 - 12 = 15 > 0$

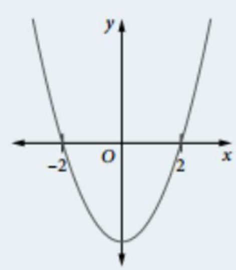
$f'(x)$ changes from +ve to −ve, so $f(x)$ has a minimum turning point at $(2, 0)$.

$\therefore$ $(-2, 32)$ is a maximum turning point and $(2, 0)$ is a minimum turning point.

(d)



In calculating the change in gradient, a sketch of $f'(x) = 3x^2 - 12$ could have been used to investigate the change in sign of $f'(x)$:



$x < -2$, $f'(x) > 0$

$x > -2$, $f'(x) < 0$: maximum turning point at $(-2, 32)$

$x < 2$, $f'(x) < 0$

$x > 2$, $f'(x) > 0$: minimum turning point at $(2, 0)$

THE FIRST DERIVATIVE AND TURNING POINTS

When selecting values of x to substitute into $f'(x)$, you need to pick values near the abscissa (x-coordinate) of the stationary point. But how near is 'near'? In Example **5** you used values that were 1 unit either side of the stationary point, and in this example that was close enough. You could have used −2.1 and −1.9 and reached the same answer, but the calculations would have been more time-consuming.

Calculations for the stationary point at (−2, 32) could have been summarised in tables as follows:

x-value	−3	−2	−1
$f'(x)$	15	0	−9
Sign of $f'(x)$	+	0	−
Direction of curve	↗	→	↘
Shape of curve			

x-value	−2.1	−2	−1.9
$f'(x)$	1.2	0	−1.2
Sign of $f'(x)$	+	0	−
Direction of curve	↗	→	↘
Shape of curve			

This would then allow you to say that the point (−2, 32) is a local maximum.

Generally the table must be presented as below:

x-value		-2	
Sign of $f'(x)$	$+$	0	$-$
Direction of curve	↗	→	↘
Shape of curve