SIMPLE HARMONIC MOTION

Consider the motion of a particle moving along a straight line back and forth, so that its displacement from a fixed point at time t is given by a sine or cosine function.

Example 5

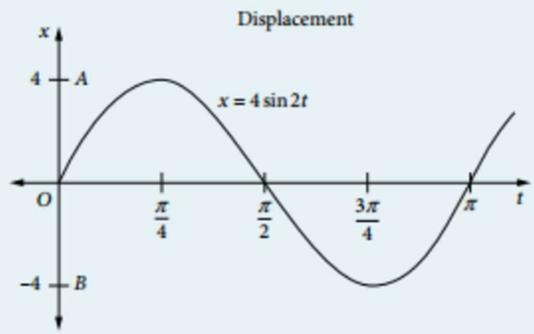
A particle moves in a straight line so that its displacement x m from a fixed point O at time t seconds is defined by $x = 4\sin 2t$. After considering properties of the graph of $x = 4\sin 2t$ to analyse the motion of the particle, find expressions for:

(a) the velocity **(b)** the acceleration. **(c)** Discuss the motion of the particle.

Solution

Look at the properties of the sine curve to help determine the nature of the movement of the particle.

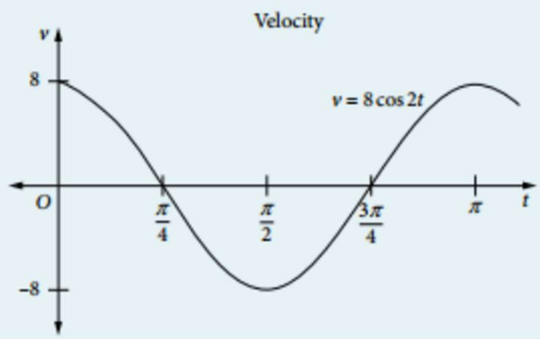
- When $t = 0$, $x = 0$
- When $t = \frac{\pi}{4}$, $x = 4$
- When $t = \frac{\pi}{2}$, $x = 0$
- When $t = \frac{3\pi}{4}$, $x = -4$
- When $t = \pi$, $x = 0$



Remember that the particle is moving in a straight line along the x-axis. It starts at $x = 0$ and takes $\frac{\pi}{4}$ seconds (approximately 0.8 s) to move to A, another $\frac{\pi}{4}$ seconds to return to O, another $\frac{\pi}{4}$ seconds to move to B and then another $\frac{\pi}{4}$ seconds to return to O again. This pattern of movement repeats every π seconds. You say that the particle **oscillates** from A to B about the point O, the centre of the motion, with a **period** of π. The distance between O and the extreme positions A and B (in this case 4 m) is called the **amplitude**.

(a) $v = \frac{dx}{dt} = 8\cos 2t$ [1]

- When $t = 0$, $v = 8$ at $x = 0$
- When $t = \frac{\pi}{4}$, $v = 0$ at $x = 4$
- When $t = \frac{\pi}{2}$, $v = -8$ at $x = 0$
- When $t = \frac{3\pi}{4}$, $v = 0$ at $x = -4$
- When $t = \pi$, $v = 8$ at $x = 0$



In the original displacement diagram above, the particle is at rest at A and B because $v = 0$ when $t = \frac{\pi}{4}$ at $x = 4$ and when $t = \frac{3\pi}{4}$ at $x = -4$.

The velocity diagram above right shows that $v = 0$ when $t = \frac{\pi}{4}$ and when $t = \frac{3\pi}{4}$.

When $t = 0$, $x = 0$ and $v = 8$. When $t = \frac{\pi}{2}$, $x = 0$ and $v = -8$. This means that at O, $v = 8$ when the particle is travelling in the positive direction (towards A) but $v = -8$ when the particle is travelling in the negative direction (towards B).

Squaring [1]:

$$v^2 = 64\cos^2 2t$$

$$v^2 = 64(1 - \sin^2 2t)$$

$$v^2 = 64\left(1 - \frac{x^2}{16}\right)$$

$$v^2 = 4(16 - x^2) \qquad [2]$$

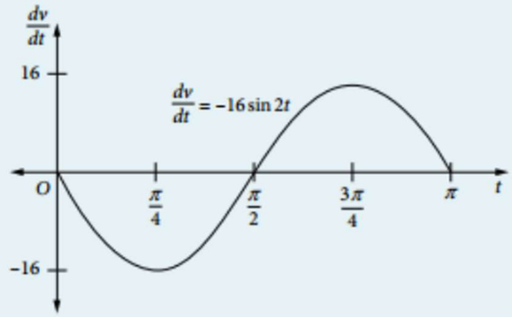
Equation [2] gives the velocity in terms of x.

(b) $\ddot{x} = \frac{dv}{dt} = -16\sin 2t$ [3]

$\ddot{x} = -4x$ [4]

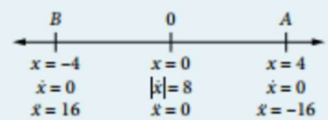
Equation [3] gives the acceleration in terms of t. Equation [4] gives the acceleration in terms of x. $\ddot{x} = -4x$ tells us that when $x = 0$, $\ddot{x} = 0$; when $x = 4$ (at A), $\ddot{x} = -16$; and when $x = -4$ (at B), $\ddot{x} = 16$. $\ddot{x}$ and x always have the opposite sign.

- When $t = 0$, $\ddot{x} = 0$ at $x = 0$
- When $t = \frac{\pi}{4}$, $\ddot{x} = -16$ at $x = 4$
- When $t = \frac{\pi}{2}$, $\ddot{x} = 0$ at $x = 0$
- When $t = \frac{3\pi}{4}$, $\ddot{x} = 16$ at $x = -4$
- When $t = \pi$, $\ddot{x} = 0$ at $x = 0$



(c) In this type of motion, the particle moves in a straight line so that its acceleration is always directed towards a fixed point in the line and the magnitude of this acceleration is proportional to its distance from the fixed point.

Taking O as the fixed point, the above description tells you that when the particle is at A ($x > 0$) its acceleration is towards O ($\ddot{x} < 0$), while when the particle is at B ($x < 0$) its acceleration is again towards O ($\ddot{x} > 0$).



Note that the equation of motion $x = 4\sin 2t$ in this example could also be written as $x = 4\cos\left(2t - \frac{\pi}{2}\right)$.

In a situation such as Example 5, $\ddot{x}$ and x are always opposite in sign and the magnitude of the acceleration is always proportional to the distance from O. This means you can take the differential equation $\ddot{x} = -4x$ from this example and write it as a general equation $\ddot{x} = -kx$, where k can be any positive constant, to define all motion of this type.

This type of motion is called **simple harmonic motion** (SHM) and it can be applied to many real-life physical situations. Because k is a positive constant, it is usually replaced by n^2 so that $\ddot{x} = -n^2 x$ is the basic equation of SHM.

In general, for simple harmonic motion you have:

• displacement x:	$x = a\cos(nt + \alpha)$, $\alpha > 0$, $n > 0$	OR	$x = a\sin(nt + \alpha)$, $\alpha > 0$, $n > 0$
• velocity, $\dot{x}$:	$\dot{x} = -an\sin(nt + \alpha)$		$\dot{x} = an\cos(nt + \alpha)$
• acceleration $\ddot{x}$:	$\ddot{x} = -an^2\cos(nt + \alpha)$		$\ddot{x} = -an^2\sin(nt + \alpha)$
or	$\ddot{x} = -n^2 x$		$\ddot{x} = -n^2 x$
• Squaring the velocity:	$v = -an\sin(nt + \alpha)$		$v = -an\cos(nt + \alpha)$
	$v^2 = a^2n^2\sin^2(nt + \alpha)$		$v^2 = a^2n^2\cos^2(nt + \alpha)$
	$= n^2\left(a^2 - a^2\cos^2(nt + \alpha)\right)$		$= n^2\left(a^2 - a^2\sin^2(nt + \alpha)\right)$
	$= n^2\left(a^2 - x^2\right)$		$= n^2\left(a^2 - x^2\right)$

So, regardless of your starting point you have the very useful result: $v^2 = n^2(a^2 - x^2)$.

As you work through this chapter, you will discover which form (sine or cosine) is best to use in a particular situation. Of course, if you start with the differential equation $\ddot{x} = -n^2 x$ and use either form, the forms will only differ by a constant, which will be given by the initial conditions.

Important results

1 When $\dot{x}=0$ (e.g. at A and B in Example 5) the magnitude of the acceleration is greatest.

2 When $\ddot{x}=0$ (i.e. at O, the centre of the motion), the speed is greatest (i.e. the velocity has its greatest or least value).

The general equation $x = a\cos(nt+\alpha)$, $\alpha > 0$, $n > 0$

If when $t=0$, $x=a$, the particle is initially at A (the extreme point) and $a = a\cos\alpha$, then $\cos\alpha = 1$ and $\alpha = 0$. The equation of motion can then be written $x = a\cos nt$.

$$x = a\cos nt \qquad [1]$$

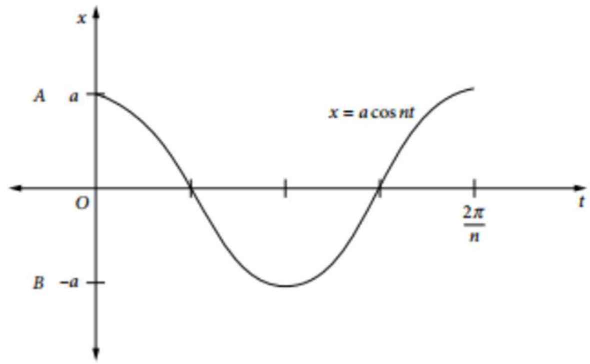
$$v = \dot{x} = -na\sin nt \qquad [2]$$

$$\begin{aligned} v^2 &= n^2a^2\sin^2 nt \\ &= n^2a^2(1-\cos^2 nt) \\ &= n^2a^2\left(1-\frac{x^2}{a^2}\right) \end{aligned}$$

$$v^2 = n^2(a^2-x^2) \qquad [3]$$

$$\frac{dv}{dt} = \ddot{x} = -n^2a\cos nt$$

$$\ddot{x} = -n^2x \qquad [4]$$



- The period T is the time for one complete oscillation: $T = \frac{2\pi}{n}$
- The frequency f is the number of oscillations per unit time: $f = \frac{1}{T} = \frac{n}{2\pi}$
- The amplitude a is the distance from the centre of motion O to either of the extreme points A or B.

Simple harmonic motion problems are usually solved using either $x = a\cos(nt+\alpha)$ or $\ddot{x} = -n^2x$, or occasionally $v^2 = n^2(a^2-x^2)$. The starting point will depend on the information given.

Simple harmonic motion (SHM)—summary

$$x = a\cos(nt+\alpha) = a\cos nt \quad \text{if } x(0)=a$$

$$v = \dot{x} = -na\sin nt$$

$$\frac{dv}{dt} = \ddot{x} = -n^2a\cos nt$$

$$\ddot{x} = -n^2x$$

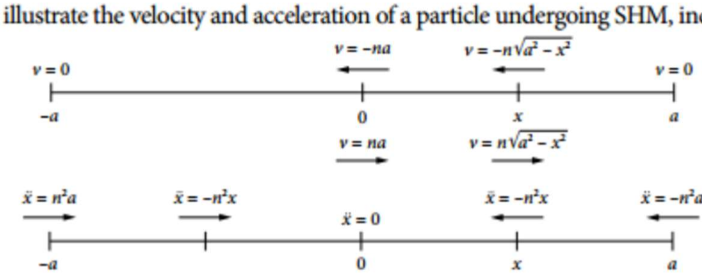
$$v^2 = n^2(a^2-x^2) \quad -a \le x \le a$$

$$T = \frac{2\pi}{n} = \frac{1}{f}$$

Note:

- $x = a\sin(nt+\alpha)$ can also describe the displacement function, but it is more conventional to use $x = a\cos(nt+\alpha)$.
- In problems involving a pendulum, the motion usually starts at the maximum displacement, so $\alpha = 0$ and the equation of motion becomes $x = a\cos nt$.

The following diagrams illustrate the velocity and acceleration of a particle undergoing SHM, including the extreme values.



Note that to the left of O, x is negative, so $-n^2x$ (the acceleration) is positive.

The equation $\ddot{x} = -n^2x$ describes the motion of a particle under the influence of a force that is directed towards the origin O and is proportional to the distance of the particle from O. The force (and hence the acceleration) is zero at O, where the speed is greatest. The magnitude of the force (and hence the acceleration) is greatest at the extreme points, where the speed is zero.

This type of motion occurs in real physical situations (either approximately or exactly) where a particle oscillates about an equilibrium position. For example:

- the to-and-fro motion of a pendulum bob
- the up-and-down motion of a mass attached to a spring
- the bobbing motion of a buoy floating on water.

Example 6

The displacement x m of a particle moving in a straight line is given by $x = 4\cos 6t$. Discuss the motion of the particle.

Solution

$x = 4\cos 6t$ is of the form $x = a\cos(nt + \alpha)$, so the motion is simple harmonic about the origin.

Amplitude $a = 4$ $\quad n = 6 \quad$ Period $= \frac{2\pi}{n} = \frac{2\pi}{6} = \frac{\pi}{3}$

When $t = 0$: $x = 4\cos 0 = 4$

$\therefore$ The particle starts 4 m to the right of O.

Velocity: $\dot{x} = -24\sin 6t$

When $t = 0$: $\dot{x} = -24\sin 0 = 0$

$\therefore$ The particle is initially at rest.

Acceleration: $\ddot{x} = -144\cos 6t$

$\ddot{x} = -36x$

When $t = 0$: $\ddot{x} = -144$

$\therefore$ The initial acceleration is 144 m s^{-2} towards O.

The motion is simple harmonic with amplitude 4 m, period $\frac{\pi}{3}$ seconds, initially at rest 4 m to the right of O with an acceleration of 144 m s^{-2} towards O.

Example 7

The motion of a particle moving along a straight line is given by the equation $\frac{d^2x}{dt^2} = -16x$.

If $x = 0$ and $v = 4$ when $t = 0$, find its displacement at any time t and state the period and amplitude.

Solution

Because $\frac{d^2x}{dt^2} = -16x = -n^2x$ where $n = 4$, the motion is simple harmonic.

$$\therefore \quad x = a\cos(nt + \alpha)$$

$$\text{For } n = 4: \quad x = a\cos(4t + \alpha) \qquad [1]$$

$$v = \dot{x} = -4a\sin(4t + \alpha) \qquad [2]$$

When $t = 0$, $x = 0$ in [1]: $0 = a\cos\alpha$

$$\alpha = \frac{\pi}{2}$$

When $t = 0$, $v = 4$ in [2]: $4 = -4a\sin\frac{\pi}{2}$

$$a = -1$$

$$\therefore \quad x = -\cos\left(4t + \frac{\pi}{2}\right)$$

You can use the identity $-\cos\theta = \cos(\pi - \theta)$ to remove the negative sign and write the answer in a more familiar form:

$$-\cos\left(4t + \frac{\pi}{2}\right) = \cos\left(\pi - 4t - \frac{\pi}{2}\right)$$

$$= \cos\left(\frac{\pi}{2} - 4t\right)$$

$$= \cos\left(4t - \frac{\pi}{2}\right)$$

$$\therefore \quad x = \cos\left(4t - \frac{\pi}{2}\right) \qquad \text{Period} = \frac{2\pi}{n} = \frac{2\pi}{4} = \frac{\pi}{2} \qquad \text{Amplitude} = a = 1$$

It is worth noticing that $\cos\left(\frac{\pi}{2} - 4t\right) = \sin 4t$, so the equation of motion could be written as $x = \sin 4t$.

This would make it easier to sketch the function.

Example 8

A particle moves in a straight line so that its acceleration at any time is given by $\ddot{x} = -4x$. Find its period, amplitude and displacement at time t given that at $t = 0$, $x = 3$ and $v = -6\sqrt{3}$.

Solution

$\ddot{x} = -4x = -n^2x$ where $n = 2$, so the motion is simple harmonic.

$$x = a\cos(nt + \alpha)$$

For $n = 2$: $\quad x = a\cos(2t + \alpha)$

When $t = 0, x = 3$: $\quad 3 = a\cos\alpha \quad [1]$

Velocity: $\quad \dot{x} = -2a\sin(2t + \alpha)$

When $t = 0, v = -6\sqrt{3}$: $\quad -6\sqrt{3} = -2a\sin\alpha$

$$3\sqrt{3} = a\sin\alpha \quad [2]$$

[2] ÷ [1]: $\quad \dfrac{a\sin\alpha}{a\cos\alpha} = \dfrac{3\sqrt{3}}{3}$

$$\tan\alpha = \sqrt{3}$$

As $a > 0$: $\sin\alpha > 0$ (from [2]) and $\cos\alpha > 0$ (from [1]), so α is in the first quadrant and is an acute angle.

Hence: $\quad \alpha = \frac{\pi}{3}$

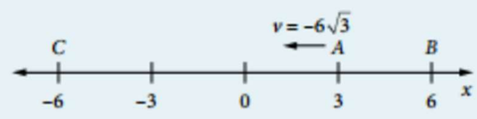
From [1]: $\quad 3 = a\cos\frac{\pi}{3}$

$$a = 6$$

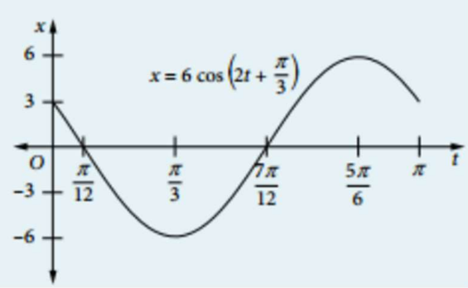
$$\therefore \quad x = 6\cos\left(2t + \frac{\pi}{3}\right)$$

Hence the period $= \frac{2\pi}{2} = \pi$, amplitude $= 6$ and displacement is given by $x = 6\cos\left(2t + \frac{\pi}{3}\right)$.

This diagram shows the motion of the particle at $t = 0$.
It starts at A and moves towards the centre of the motion.
C and B are the extreme points of the motion.



This diagram shows the displacement x for any time t in the domain $0 \le t \le \pi$.



Example 9

The speed v m s^{-1} of a particle moving in a straight line is given by $v^2 = 6 + 4x - 2x^2$, where the magnitude of its displacement from a fixed point O is x m. Show that the motion is simple harmonic and find:

(a) the centre of the motion **(b)** the period **(c)** the amplitude.

Solution

$v^2 = f(x)$, so it seems likely that using $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ might help.

$$v^2 = 6 + 4x - 2x^2$$

$$\tfrac{1}{2}v^2 = 3 + 2x - x^2$$

Differentiate with respect to x: $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2 - 2x$

Hence: $\ddot{x} = -2(x - 1)$

With $y = x - 1$: $\ddot{y} = -2y$ (as $\ddot{y} = \ddot{x}$)

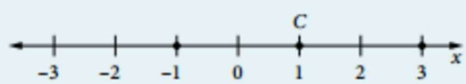
This is simple harmonic motion about $y = 0$.

(a) With $y = x - 1$, SHM about $y = 0$ is the same as SHM about $x = 1$.

(b) $n^2 = 2$, so $n = \sqrt{2}$ and the period $= \frac{2\pi}{\sqrt{2}} = \pi\sqrt{2}$.

(c) The extreme positions of the particle are found where $v = 0$.

$$6 + 4x - 2x^2 = 0$$
$$-2(x^2 - 2x - 3) = 0$$
$$(x + 1)(x - 3) = 0$$
$$x = -1 \text{ or } 3$$



The particle oscillates between $x = -1$ and $x = 3$ about the centre $x = 1$, so the amplitude is 2 m.

Example 14

A particle is moving with SHM about the point $x = 5$ cm. The particle starts from rest at the point $x = 14$ cm with a period of 4π seconds. Calculate:

(a) the amplitude **(b)** the acceleration when $t = 4$.

Solution

(a) The centre of motion is $x = 5$ so the displacement is given by: $x = a\cos(nt + \alpha) + 5$

$t = 0$, $x = 14$: $14 = a\cos\alpha + 5$

$$a\cos\alpha = 9$$

Velocity is given by: $\dot{x} = -an\sin(nt + \alpha)$

$t = 0$, $\dot{x} = 0$: $0 = -an\sin\alpha$

$$\alpha = 0$$
$$a\cos 0 = 9$$
$$a = 9$$

The amplitude is 9.

(b) Period: $4\pi = \frac{2\pi}{n}$

$$n = \frac{1}{2}$$

$$\dot{x} = -9n\sin nt$$

$n = \frac{1}{2}$: $\dot{x} = -\frac{9}{2}\sin\frac{t}{2}$

$$\ddot{x} = -\frac{9}{4}\cos\frac{t}{2}$$

$t = 4$: $\ddot{x} = -\frac{9}{4}\cos 2$

$$\approx 0.936 \text{ cm s}^{-2}$$

Example 10

A particle moves in a straight line so that its position at any time t is given by $x = 3\cos 2t + 4\sin 2t$.

(a) Show that $3\cos 2t + 4\sin 2t = 5\cos(2t - \alpha)$ where $0 < \alpha < \frac{\pi}{2}$ and $\tan\alpha = \frac{4}{3}$.

(b) Show that the motion is simple harmonic and find its greatest speed in metres per second.

Solution

(a)

$$\text{Expression} = 3\cos 2t + 4\sin 2t$$

$$\sqrt{3^2+4^2} = 5: \quad = 5\left(\tfrac{3}{5}\cos 2t + \tfrac{4}{5}\sin 2t\right)$$

$$\cos\alpha = \tfrac{3}{5},\ \sin\alpha = \tfrac{4}{5}: \quad = 5(\cos 2t\cos\alpha + \sin 2t\sin\alpha)$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{4}{3}: \quad = 5\cos(2t - \alpha) \text{ where } 0 < \alpha < \tfrac{\pi}{2} \text{ and } \tan\alpha = \tfrac{4}{3}$$

(b)

$$x = 5\cos(2t - \alpha)$$

Velocity: $\dot{x} = -10\sin(2t - \alpha)$

Acceleration: $\ddot{x} = -20\cos(2t - \alpha)$

$= -4x$

$\ddot{x} = -n^2 x$ with $n = 2$, so the motion is simple harmonic.

The greatest speed occurs when $\ddot{x} = 0$, i.e. when $\cos(2t - \alpha) = 0$.

Hence: $2t - \alpha = \frac{\pi}{2}$

Thus: $\dot{x} = -10\sin\frac{\pi}{2}$

$= -10$ The greatest speed is $10\,\text{m s}^{-1}$.

Example 11

A particle moving in a straight line with SHM has a speed of $15\,\text{m s}^{-1}$ when passing through its mean position (the centre of the motion). Find the amplitude of the motion and the acceleration in the extreme positions, given that the period of the motion is 2 seconds.

Solution

$x = 0 \qquad |v| = 15 \qquad T = 2 = \frac{2\pi}{n}$ hence $n = \pi$.

The values of x, v and n are known. Use the result $v^2 = n^2(a^2 - x^2)$ to find a.

$$225 = \pi^2(a^2 - 0)$$

$$a^2 = \frac{225}{\pi^2}$$

$$a = \frac{15}{\pi} \quad \text{as } a > 0$$

Motion is SHM, so: $\ddot{x} = -n^2 x$

Hence: $\ddot{x} = -\pi^2 x$

At the extreme position, $x = a = \frac{15}{\pi}$: $\ddot{x} = -\pi^2 \times \frac{15}{\pi} = -15\pi$

At the other extreme position, $x = -\frac{15}{\pi}$: $\ddot{x} = -\pi^2 \times \left(-\frac{15}{\pi}\right) = 15\pi$

Thus the amplitude of the motion is $\frac{15}{\pi}$ metres and the magnitude of the acceleration is $15\pi\,\text{m s}^{-2}$.

Example 12

A particle is moving in a straight line with SHM. The velocity of the particle is respectively $\sqrt{5}\,\text{m s}^{-1}$ and $2\,\text{m s}^{-1}$ at distances of 1 m and 2 m from the centre of motion. Find:

(a) the length of the path **(b)** the period of the motion.

Solution

(a) Given $x = 1, v = \sqrt{5}; x = 2, v = 2$.

Hence use: $v^2 = n^2(a^2 - x^2)$

At $x = 1, v = \sqrt{5}$: $5 = n^2(a^2 - 1)$ [1]

At $x = 2, v = 2$: $4 = n^2(a^2 - 4)$ [2]

[1] ÷ [2]: $\frac{5}{4} = \frac{n^2(a^2 - 1)}{n^2(a^2 - 4)}$

$$5a^2 - 20 = 4a^2 - 4$$

$$a^2 = 16$$

$$a = 4 \quad \text{as } a > 0$$

The amplitude of the motion is 4 m, so the length of the path ($2a$) is 8 m.

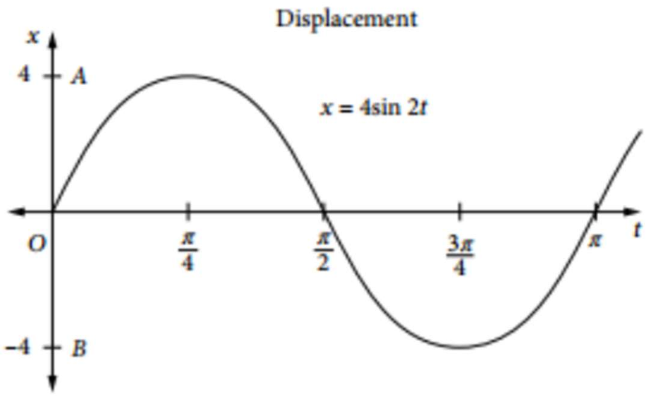
(b) Substitute $a = 4$ into [1]: $5 = 15n^2$

$$\therefore \quad n = \frac{1}{\sqrt{3}} \quad \text{as } n > 0$$

Thus the period of the motion is $T = \frac{2\pi}{n} = 2\pi\sqrt{3}$ seconds.

Simple harmonic motion about the point $x = c$

In Example 5, you considered a particle moving in a straight line with displacement x m from a fixed point O at time t seconds defined by the equation $x = 4\sin 2t$. The graph of the displacement is given in the following diagram.



Simple harmonic motion about the point $x = c$ will involve a vertical translation of the trigonometric graph. This will be considered in the following example.

Example 13

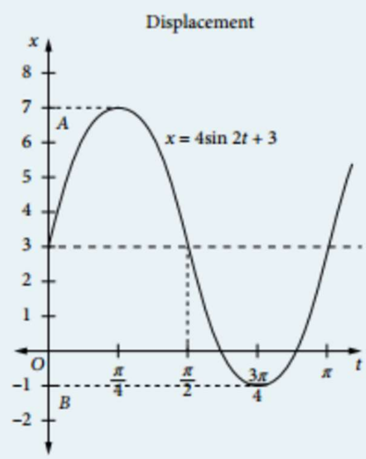
The equation of motion changes to $x = 4\sin 2t + 3$. By considering the graph of this function, analyse the motion of the particle, and find expressions for **(a)** velocity, **(b)** the acceleration. **(c)** Discuss the motion of the particle.

Solution

The effect of the +3 on the equation $x = 4\sin 2t$ is to translate the graph 3 units upwards with no horizontal translation.

The graph shows the following points.

- When $t = 0$, $x = 3$.
- When $t = \frac{\pi}{4}$, $x = 7$.
- When $t = \frac{\pi}{2}$, $x = 3$.
- When $t = \frac{3\pi}{4}$, $x = -1$.
- When $t = \pi$, $x = 3$.

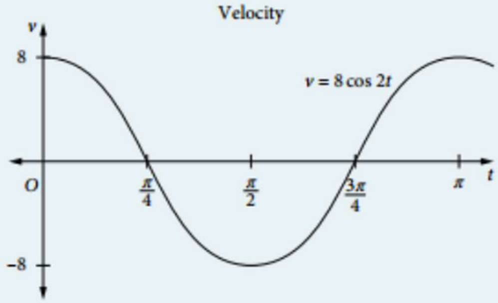


The particle is moving in a straight line along the x-axis. It starts at $x = 3$ and takes $\frac{\pi}{4}$ seconds to move to A, a distance of 4 m. It takes another $\frac{\pi}{4}$ seconds to return to $x = 3$ and then another $\frac{\pi}{4}$ seconds to move to B, a point 4 m below $x = 3$. It then takes $\frac{\pi}{4}$ seconds to return to $x = 3$.

The pattern repeats every π seconds and the particle oscillates about $x = 3$, the centre of the motion, with a period of π. The amplitude of the motion is 4.

(a) $x = 4\sin 2t + 3$

$\frac{dx}{dt} = 8\cos 2t$



This is the same as the velocity function in Example 5 (page 172), so shifting the centre of the motion makes no change to the velocity of the motion as a function of time.

The particle is instantaneously at rest at A and B because $v = 0$ when $t = \frac{\pi}{4}$ and $t = \frac{3\pi}{4}$.

When $t = 0$, $x = 3$, $v = 8$ so the particle is travelling towards A with its greatest velocity.

When $t = \frac{\pi}{2}$, $x = 3$, $v = -8$ so the particle is travelling towards B with its least velocity.

Squaring the velocity function gives $v^2 = 64\cos^2 2t$

$$v^2 = 64\left(1 - \sin^2 2t\right)$$

But $\sin 2t = \frac{x-3}{4}$: $v^2 = 64\left(1 - \left(\frac{x-3}{4}\right)^2\right)$

$$v^2 = 4\left(16 - (x-3)^2\right)$$

This gives the velocity in terms of x and reflects the shift of 3 units in the centre of the motion.

(b) $\dot{x} = v = 8\cos 2t$

$\ddot{x} = \frac{dv}{dt} = -16\sin 2t$

This is the same as the acceleration function in Example 5 (page 172), so shifting the centre of the motion makes no change to the acceleration of the motion as a function of time.

But $\sin 2t = \frac{x-3}{4}$: $\ddot{x} = -16 \times \frac{x-3}{4}$

$$\ddot{x} = -4(x-3)$$

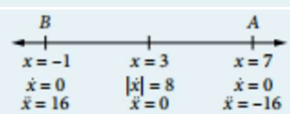
This gives the acceleration in terms of x and reflects the shift of 3 units in the centre of the motion. It shows that the acceleration is always directed towards the centre of motion.

Thus when $x = 3$, $\ddot{x} = 0$: the acceleration is zero when the particle passes through the centre of the motion.

When $x = 7$, $\ddot{x} = -16$: the acceleration is least when the displacement is greatest.

When $x = -1$, $\ddot{x} = 16$: the acceleration is greatest when the displacement is least.

(c) In this motion, the particle moves in a straight line so that its acceleration is always directed towards the fixed point in the line that is the centre of the motion. The magnitude of this acceleration is proportional to its distance from the centre of the motion.



$x = 3$ is the centre of the motion, when the particle is at A ($x > 3$) its acceleration is towards $x = 3$ ($\ddot{x} < 0$), while when the particle is at B ($x < 3$) its acceleration is again towards $x = 3$ ($\ddot{x} > 0$).

Thus the equations of motion for simple harmonic motion about the point $x = c$ may be written as:

Displacement:	$x = a\sin(nt + \alpha) + c$	$x = a\cos(nt + \alpha) + c$
Velocity:	$\dot{x} = an\cos(nt + \alpha)$	$\dot{x} = -an\sin(nt + \alpha)$
Acceleration:	$\ddot{x} = -an^2\sin(nt + \alpha)$	$\ddot{x} = -an^2\cos(nt + \alpha)$
or:	$\ddot{x} = -n^2(x - c)$	

where $x = c$ is the centre of the motion, a is the amplitude, $\frac{2\pi}{n}$ is the period and $\frac{\alpha}{n}$ the phase shift.

Example 15

Assume that the tides of the ocean rise and fall in SHM. The depth of water y metres in a harbour channel is given by $y = 10 - 2\cos\frac{\pi t}{6}$ where t is the number of hours after low tide. On a particular day, low tide occurs at 7 a.m. when the channel is 8 metres deep. On the same day, high tide occurs at 1 p.m. when the channel is 12 metres deep.

(a) Show that these values satisfy the equation $y = 10 - 2\cos\frac{\pi t}{6}$.

(b) What is the depth of the channel at 10 a.m.?

(c) State the amplitude and period of the motion.

(d) Draw the graph of $y = 10 - 2\cos\frac{\pi t}{6}$ for $0 \le t \le 12$.

(e) A ship needs at least 9 metres of water in the channel to be able to pass through safely. Use your graph to find when the ship can safely pass through the channel.

(f) Find the answer to part **(e)** algebraically.

Solution

(a) At 7 a.m., $t = 0$:

$y = 10 - 2\cos(0) = 10 - 2 = 8\text{ m}$

The depth at 7 a.m. is 8 metres.

At 1 p.m., $t = 6$:

$y = 10 - 2\cos(\pi) = 10 - 2\times(-1) = 10 + 2 = 12\text{m}$

The depth at 1 p.m. is 12 metres.

(b) 10 a.m., $t = 3$:

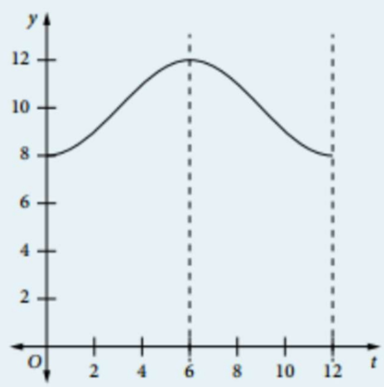
$y = 10 - 2\cos\left(\frac{\pi}{2}\right) = 10 - 0 = 10\text{ m}$

The depth at 10 a.m. is 10 metres.

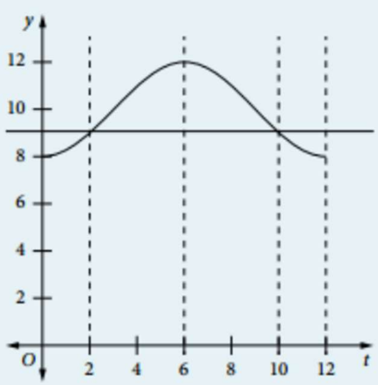
(This is halfway between low tide and high tide.)

(c) Amplitude = 2 m; period = 12 hours

(d)



(e) Draw the line $y = 9$ on the graph. When the curve is above the line it is safe to pass through the channel.



It is safe to pass through the channel from 2 hours after low tide (i.e. after 9 a.m.) until 10 hours after low tide (i.e. before 5 p.m.).

(f) To find the answer algebraically, solve $10 - 2\cos\left(\frac{\pi t}{6}\right) \ge 9$ as an equation and interpret the result:

$$2\cos\left(\frac{\pi t}{6}\right) = 1$$

$$\cos\left(\frac{\pi t}{6}\right) = \frac{1}{2}$$

$$\frac{\pi t}{6} = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$\therefore \quad t = 2, 10$$