DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

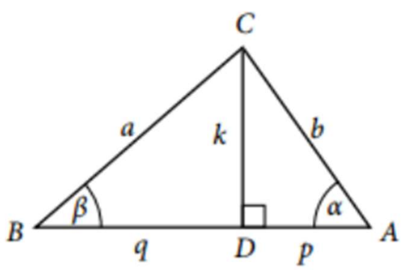
In the previous lesson, we proved that $\lim_{\theta \to 0} \left(\frac{\sin\theta}{\theta} \right) = 1$ and that $\lim_{\theta \to 0} \left(\frac{1 - \cos}{\theta^2} \right) = \frac{1}{2}$

To derive the derivatives of trigonometric functions, we also need another result, that was demonstrated in the Year 11 Extension 1 course, namely:

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$$

Proof:

Consider the triangle ABC



$$Area\ \Delta ABC = \frac{1}{2} ab \sin C$$

$$Area\ \Delta ABC = \frac{1}{2} ab \sin[\pi - (\alpha + \beta)]$$

$$Area\ \Delta ABC = \frac{1}{2} ab \sin(\alpha + \beta)$$

as $\sin(\pi - x) = \sin x$

But the area of triangle ABC can also be calculated through another way, i.e.:

$$Area\ \Delta ABC = \frac{1}{2} kp + \frac{1}{2} kq$$

Hence:

$$\frac{1}{2} ab \sin(\alpha + \beta) = \frac{1}{2} kp + \frac{1}{2} kq$$

$$\Leftrightarrow ab \sin(\alpha + \beta) = kp + kq$$

$$\Leftrightarrow \sin(\alpha + \beta) = \frac{kp}{ab} + \frac{kq}{ab}$$

$$\Leftrightarrow \sin(\alpha + \beta) = \frac{k}{a} \times \frac{p}{b} + \frac{k}{b} \times \frac{q}{a}$$

$$\Leftrightarrow \sin(\alpha + \beta) = \sin\alpha \times \cos\beta + \sin\beta \times \cos\beta$$

which is what was to be demonstrated.

Equipped with these two results [namely 1) calculation of limits and 2) the sine of a sum of angles], we can now differentiate $f(x) = sin\,x$, using first principle of differentiation.

Note that you won't need to remember this demonstration in the Maths Advanced course, however you will need to know the derivatives of trigonometric functions.

Derivative of $f(x) = sin\,x$

$$\frac{df(x)}{dx} = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$

which is the definition of the derivative function of $f(x)$. Applying that to $f(x) = sin\,x$:

$$\frac{d(sin\,x)}{dx} = \lim_{h \to 0}\left(\frac{sin(x+h) - sin\,x}{h}\right)$$

$$\frac{d(sin\,x)}{dx} = \lim_{h \to 0}\left(\frac{sin\,x\,cos\,h + cos\,x\,sin\,h - sin\,x}{h}\right)$$

$$\frac{d(sin\,x)}{dx} = \lim_{h \to 0}\left(\frac{sin\,x\,(cos\,h - 1) + cos\,x\,sin\,h}{h}\right)$$

$$\frac{d(sin\,x)}{dx} = \lim_{h \to 0}\left(\frac{sin\,x\,(cos\,h - 1)}{h}\right) + \lim_{h \to 0}\left(\frac{cos\,x\,sin\,h}{h}\right)$$

$$\frac{d(sin\,x)}{dx} = sin\,x \times \lim_{h \to 0}\left(\frac{cos\,h - 1}{h}\right) + cos\,x \times \lim_{h \to 0}\left(\frac{sin\,h}{h}\right)$$

$$\frac{d(sin\,x)}{dx} = sin\,x \times \lim_{h \to 0} h \times \left(\frac{cos\,h - 1}{h^2}\right) + cos\,x \times \underbrace{\lim_{h \to 0}\left(\frac{sin\,h}{h}\right)}_{=1}$$

$$\frac{d(sin\,x)}{dx} = sin\,x \times \lim_{h \to 0} h \times \underbrace{\left(\frac{cos\,h - 1}{h^2}\right)}_{tends\ towards\ \frac{1}{2}} + cos\,x \times 1$$

$$\frac{d(sin\,x)}{dx} = sin\,x \times \underbrace{\lim_{h \to 0} h \times \left(\frac{cos\,h - 1}{h^2}\right)}_{tends\ towards\ 0\ as\ h \to 0} + cos\,x$$

Therefore $$\frac{d(sin\,x)}{dx} = cos\,x$$

DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

Derivative of $\boldsymbol{f(x) = cos\, x}$

The derivative of $f(x) = cos\, x$ is derived more simply using the equality $cos\, x = sin\left(\frac{\pi}{2} - x\right)$

We use the chain rule: $\left[f\big(g(x)\big)\right]' = f'\big(g(x)\big) \times g'(x)$

So: $(cos\, x)' = \left(sin\left(\frac{\pi}{2} - x\right)\right)' = cos\left(\frac{\pi}{2} - x\right) \times (-1)$

But: $cos\left(\frac{\pi}{2} - x\right) = sin\, x$

Therefore $(cos\, x)' = -\, sin\, x$

Derivative of $\boldsymbol{f(x) = tan\, x}$

As $tan\, x = \frac{\sin x}{\cos x} = \frac{u(x)}{v(x)}$ we use the quotient rule, i.e. $\frac{d(^u/_v)}{dx} = \frac{u'v - v'u}{v^2}$

$$\frac{d(tan\, x)}{dx} = \frac{cos\, x \times cos\, x - (-\, sin\, x) \times sin\, x}{cos^2\, x}$$

$$\frac{d(tan\, x)}{dx} = \frac{cos^2\, x + sin^2\, x}{cos^2\, x}$$

$$\frac{d(tan\, x)}{dx} = \frac{1}{cos^2\, x}$$

$$\frac{d(tan\, x)}{dx} = sec^2\, x$$

as $sec\, x = \frac{1}{cos}$

Derivatives of $\boldsymbol{sin\big(f(x)\big), cos\big(f(x)\big)}$ and $\boldsymbol{tan\big(f(x)\big)}$

These are found using the chain rule:

$$\left[sin\big(f(x)\big)\right]' = cos\big(f(x)\big) \times f'(x)$$

$$\left[cos\big(f(x)\big)\right]' = -sin\big(f(x)\big) \times f'(x)$$

$$\left[tan\big(f(x)\big)\right]' = sec^2\big(f(x)\big) \times f'(x)$$

The derivatives of the reciprocal trigonometric functions $\sec x = \frac{1}{\cos x}$,of $\csc x = \frac{1}{\sin x}$ and $\cot x = \frac{1}{\tan x}$ are also found using the chain rule.

Example 3

Differentiate: **(a)** $f(x) = \sin(2x+3)$ **(b)** $f(x) = 5\cos x^2$ **(c)** $f(x) = 3\tan 4x$

Solution

(a)
$$f(x) = \sin(2x+3)$$
$$f'(x) = 2\cos(2x+3)$$

(b)
$$f(x) = 5\cos x^2$$
$$f'(x) = -5\sin x^2 \times 2x = -10x\sin x^2$$

(c)
$$f(x) = 3\tan 4x$$
$$f'(x) = 3 \times 4\sec^2(4x) = 12\sec^2 4x$$

Example 4

Differentiate: **(a)** $f(x) = \sin^2 x$ **(b)** $f(x) = \sin x \cos x$ **(c)** $f(x) = \cos \pi x - \tan \pi x$

Solution

(a)
$$f(x) = \sin^2 x$$
$$f(x) = (\sin x)^2$$
$$f'(x) = 2\sin x \cos x$$

(b)
$$f(x) = \sin x \cos x$$
$$f'(x) = \cos x \cos x + \sin x(-\sin x) = \cos^2 x - \sin^2 x$$

(c)
$$f(x) = \cos \pi x - \tan \pi x$$
$$f'(x) = -\pi \sin \pi x - \pi \sec^2 \pi x = -\pi(\sin \pi x + \sec^2 \pi x)$$

Example 5

Differentiate: **(a)** $f(x) = \dfrac{\sin(3x-1)}{\cos x}$ **(b)** $f(x) = \dfrac{e^x}{\sin x}$

Solution

(a)
$$f(x) = \frac{\sin(3x-1)}{\cos x}$$
$$f'(x) = \frac{3\cos(3x-1)\cos x - \sin(3x-1)(-\sin x)}{\cos^2 x} = \frac{3\cos(3x-1)\cos x + \sin(3x-1)\sin x}{\cos^2 x}$$

(b)
$$f(x) = \frac{e^x}{\sin x}$$
$$f'(x) = \frac{e^x \sin x - e^x \cos x}{\sin^2 x} = \frac{e^x(\sin x - \cos x)}{\sin^2 x}$$

Example 6

Differentiate: **(a)** $f(x) = \sin(3x-1)\sec x$ **(b)** $e^x \operatorname{cosec} x$ **(c)** $x^2 \cot x$

Solution

(a)
$$f(x) = \sin(3x-1)\sec x$$
$$f'(x) = 3\sec x\cos(3x-1) + \sin(3x-1)\sec x \tan x = \sec x(3\cos(3x-1) + \sin(3x-1)\tan x)$$
or
$$\frac{3\cos(3x-1)\cos x + \sin(3x-1)\sin x}{\cos^2 x}$$

(b)
$$f(x) = e^x \operatorname{cosec} x$$
$$f'(x) = e^x \operatorname{cosec} x - e^x \operatorname{cosec} x \cot x = e^x \operatorname{cosec} x(1 - \cot x)$$
or
$$\frac{e^x(\sin x - \cos x)}{\sin^2 x}$$

(c)
$$f(x) = x^2 \cot x$$
$$f'(x) = 2x\cot x - x^2 \operatorname{cosec}^2 x = \frac{x(2\sin x\cos x - x)}{\sin^2 x}$$
or
$$\frac{x\left(2\tan x - x\sec^2 x\right)}{\tan^2 x}$$

Note: With reciprocal trigonometric function ratios there may be more than one simplest form of an expression.