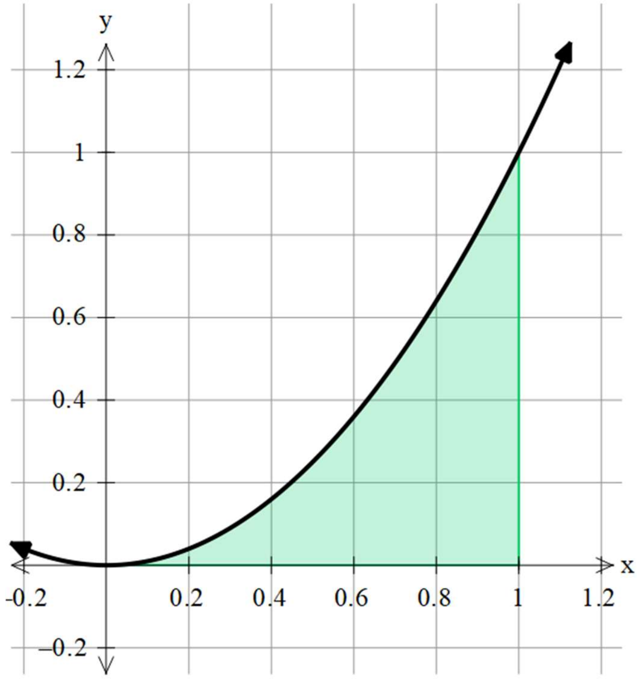
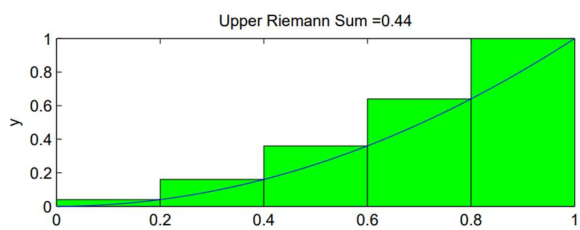
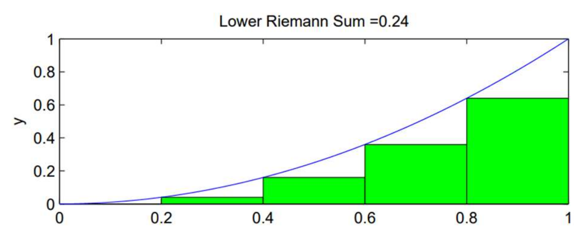
AREA UNDER A CURVE

This is the graph of $y = x^2$; the area underneath the curve between $x = 0$ and $x = 1$ has been shaded.

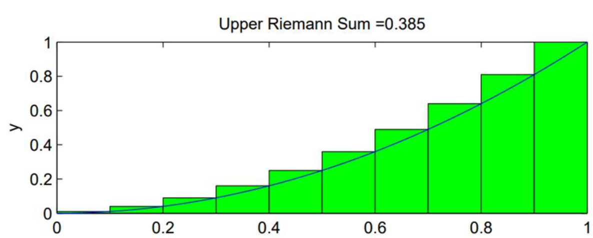


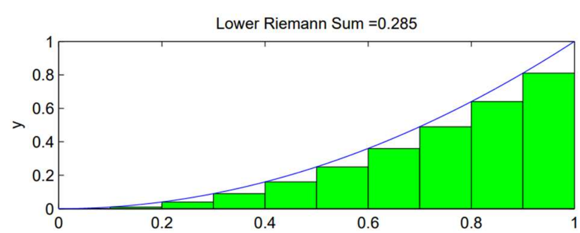
Now if we subdivide the x-coordinates into 5 equal parts, we can get an upper estimate and a lower estimate; these estimates are called Riemann Sums[1] (<u>note that in the graphs below, the vertical scale has been reduced</u>).





If we subdivide the x-coordinates into 10 equal parts, we can get an upper estimate or a lower estimate, both of them more accurate than when the x-coordinates were divided into 5 parts.

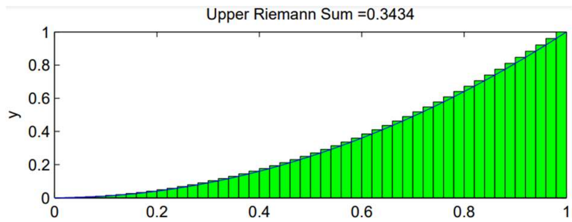


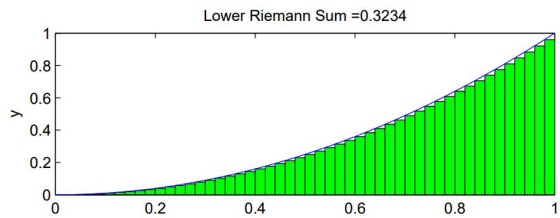


[1] from the German mathematician Georg Friedrich Bernhard Riemann (1826-1866)

AREA UNDER A CURVE

If we subdivide the x-coordinates into **50** equal parts, we can get an upper estimate or a lower estimate, both of them more accurate than when the x-coordinates were divided into 10 parts.

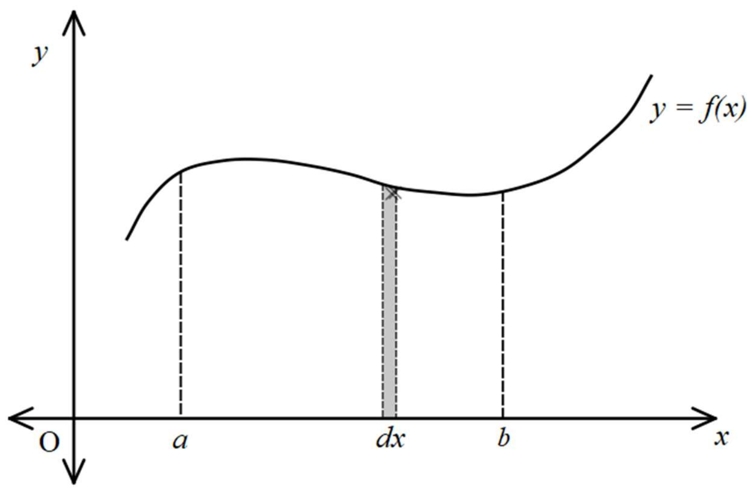




In the last case, we see that the area underneath the graph has now been shown to be between 0.3234 and 0.3434.

Now if we were going to subdivide further, the estimates would become closer and closer to the real value of the area (which we will prove later to be exactly 1/3).

It was then thought that to represent the area underneath the curve $y = f(x)$, we could divide it in an infinite number of small rectangles of infinitesimal width dx and of height $f(x)$, i.e. each of area $f(x)\ dx$, as shown below:



The area underneath the curve $y = f(x)$ between $x = a$ and $x = b$ would be the **Sum of all these infinitesimally small rectangles**, and so the notation adopted for this total area was:

$$\int_a^b f(x)\ dx$$

This is known as "the definite integral of the function $f(x)$ between $x = a$ and $x = b$".

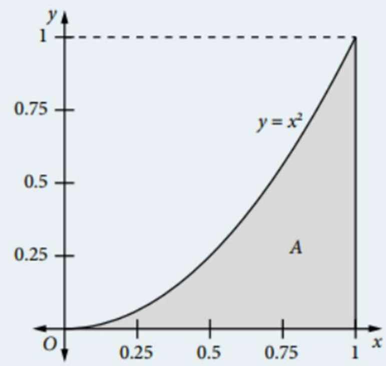
Example 2

Find an approximation of the area of the region bounded by the curve $y = x^2$, the x-axis and the ordinates at $x = 0$ and $x = 1$, using rectangles with:

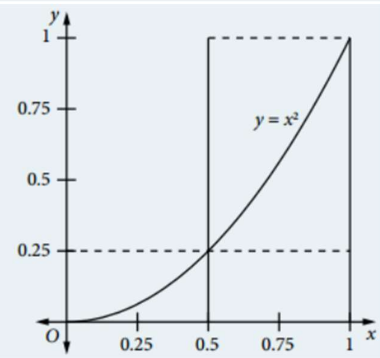
(a) one subinterval **(b)** two subintervals **(c)** four subintervals.

Solution

(a) Draw a diagram showing the region:
For $0 \le x \le 1$ and one subinterval,
$x = 0$, $y = 0$, is the smallest value;
$x = 1$, $y = 1$, is the largest value.
Hence $0 < A < 1$

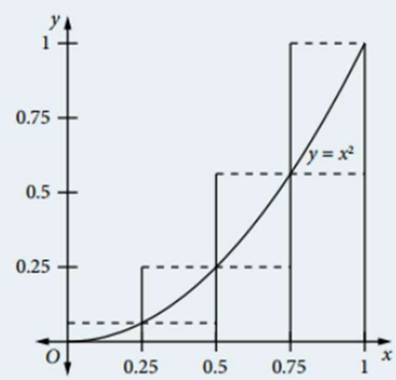


(b) For $0 \le x \le 1$ and two subintervals:
For $0 \le x \le 0.5$:
$x = 0$, $y = 0$ is smallest value; $x = 0.5$, $y = 0.25$ is largest value.
For $0.5 \le x \le 1$:
$x = 0.5$, $y = 0.25$ is smallest value; $x = 1$, $y = 1$ is largest value.
Hence: $(0 + 0.25) \times 0.5 \le A \le (0.25 + 1) \times 0.5$
$0.125 \le A \le 0.625$



(c) For $0 \le x \le 1$ and four subintervals:

$0 \le x \le 0.25$: $x = 0$, $y = 0$ is smallest value
$x = 0.25$, $y = 0.0625$ is largest value
$0.25 \le x \le 0.5$: $x = 0.25$, $y = 0.0625$ is smallest value
$x = 0.5$, $y = 0.25$ is largest value
$0.5 \le x \le 0.75$: $x = 0.5$, $y = 0.25$ is smallest value
$x = 0.75$, $y = 0.5625$ is largest value
$0.75 \le x \le 1$: $x = 0.75$, $y = 0.5625$ is smallest value
$x = 1$, $y = 1$ is largest value



Hence: $(0 + 0.0625 + 0.25 + 0.5625) \times 0.25 \le A \le (0.0625 + 0.25 + 0.5625 + 1) \times 0.25$
$0.218\,75 \le A \le 0.468\,75$

The area is between 0.218 75 and 0.468 75. Taking the average of these two values gives an approximate area of 0.343 75.