FINANCIAL MATHEMATICS - CHAPTER REVIEW

1 Are the following either arithmetic or geometric sequences? Explain.

(a) $2, -6, 10, -14, \ldots$ (b) $4+\sqrt{3},\ 1+3\sqrt{3},\ -2+5\sqrt{3},\ -5+7\sqrt{3},\ \ldots$ (c) $1.6, 2.4, 3.6, 5.4, \ldots$

a) $-6-2=-8$ whereas $10+6=16$ so not arimethic ; $\frac{-6}{2}=-3$ whereas $\frac{10}{-6}=-\frac{5}{3}$ so neither

b) $1+3\sqrt{3}-(4+\sqrt{3})=-3+2\sqrt{3}$ whereas $-2+5\sqrt{3}-(1+3\sqrt{3})=-3+2\sqrt{3}$ so arithmetic.

c) $\frac{2.4}{1.6}=1.5$ whereas $\frac{3.6}{2.4}=1.5$ and $\frac{5.4}{3.6}=1.5$ so geometric

2 For the arithmetic sequence 22, 15, 8, 1, … find:

(a) the 10th term (b) the value of k if $T_k=-90$.

a) 1st term is 22, common difference is -7 so $T_n=22+(n-1)(-7)$

So for $n=10$ $T_{10}=22+(10-1)\times(-7)=-41$

b) $-90=22+(k-1)(-7)$ so $(k-1)(-7)=-112$ so $k-1=16$

so $k=17$

3 For the geometric sequence 36, 126, 441, 1543.5, … find:

(a) the 7th term (b) the smallest value of k for which $T_k>1\,000\,000$.

1st term is 36, common ratio is $\frac{126}{36}=\frac{7}{2}=3.5$

a) $T_n=36\times3.5^{n-1}$ so for $n=7$, $T_7=36\times3.5^6=66177.5625$

b) $36\times\left(\frac{7}{2}\right)^{n-1}>1{,}000{,}000 \iff \left(\frac{7}{2}\right)^{n-1}>\frac{250{,}000}{9}$

$\Rightarrow \ln\left(\frac{7}{2}\right)^{n-1}>\ln\left(\frac{250{,}000}{9}\right) \iff (n-1)\ln\left(\frac{7}{2}\right)>\ln\left(\frac{250{,}000}{9}\right)$

so $n-1>\frac{\ln(250{,}000)-\ln 9}{\ln 7-\ln 2}$ $\boxed{n>\underbrace{\frac{\ln(250{,}000)-\ln 9}{\ln 7-\ln 2}+1}_{\simeq 9.2}}$

$n>10$

FINANCIAL MATHEMATICS - CHAPTER REVIEW

4 If $T_4 = 600$ and $T_{10} = 75$ are two terms of a sequence. Find the first three terms if:

(a) the sequence is arithmetic (b) the sequence is geometric.

a) $T_4 = T_1 + (n-1)d = T_1 + 3d = 600$ and $T_{10} = T_1 + 9d = 75$

By elimination, $6d = 75 - 600 = -525$ so $d = -87.5$ and $T_1 = 862.5$

So $T_2 = 775$ and $T_3 = 687.5$

b) $T_4 = T_1 \times r^{4-1} = T_1 r^3 = 600$

whereas $T_{10} = T_1 \times r^{10-1} = T_1 r^9 = 75$

Dividing the two, we obtain $r^6 = \frac{75}{600} = \frac{1}{8}$ so $r^2 = \frac{1}{2}$ $r = \pm\frac{1}{\sqrt{2}}$

if $r = 1/\sqrt{2}$ then $T_1 = 600 \times (\sqrt{2})^3 = 1200\sqrt{2}$ $T_2 = 1200$ $T_3 = 600\sqrt{2}$

if $r = -\frac{1}{\sqrt{2}}$ then $T_1 = 600(-\sqrt{2})^3 = -1200\sqrt{2}$ $T_2 = 1200$

$T_3 = -600\sqrt{2}$

6 Evaluate $6 + 3 + 1.5 + \ldots = 6\left(1 + \frac{1}{2} + \frac{1}{4} + \ldots\right)$

$\frac{3}{6} = \frac{1}{2}$ whereas $\frac{1.5}{3} = \frac{1}{2}$ so geometric series, of 1st term 6 and common ratio ½

$$S = \frac{6}{1-r} = \frac{6}{1-\frac{1}{2}} = \frac{6}{\frac{1}{2}} = 12$$

7 A ball is dropped from a height of 20 m and rebounds to a height of 18 m. If every time it rebounds it rises to nine-tenths of its previous height, calculate the total number of metres it could travel.

$20,\ 18,\ 18\times\frac{9}{10} = 16.2,\ 16.2\times\frac{9}{10},\ \dots$

$S = 20 + 2\times 18 + 2\times 16.2 + 2\times(16.2\times 0.9) + \dots$

$S = 20 + 2\times 18 + 2\times(18\times 0.9) + 2\times(18\times 0.9^2) + \dots$

$S = 20 + 2\times 18\left[1 + 0.9 + 0.9^2 + \dots\right]$

$S = 20 + 36\times\dfrac{1}{1-0.9} = 380\text{ m}$

8 Three numbers whose sum is 15 are successive terms of an arithmetic series. If 1, 1 and 4 are added to these three numbers respectively, the resulting numbers are successive terms of a geometric series. Find the numbers.

$T_i + T_{i+1} + T_{i+2} = 15$ and further $T_i + (T_i + d) + (T_i + 2d) = 15$

$\Leftrightarrow 3T_i + 3d = 15$

so $T_i + d = 5$

Further $\dfrac{T_{i+1}+1}{T_i+1} = \dfrac{T_{i+2}+4}{T_{i+1}+1}$

So $\dfrac{T_i+d+1}{T_i+1} = \dfrac{T_i+2d+4}{T_i+d+1}$

$\Leftrightarrow \dfrac{5+1}{5-d+1} = \dfrac{5-d+2d+4}{5+1}$

$\Leftrightarrow 36 = (9+d)(6-d) \Leftrightarrow -d^2 - 3d + 18 = 0$

or $d^2 + 3d - 18 = 0$

$\Delta = 81 = 9^2$

$d = \dfrac{-3+9}{2} = 3$ or $d = -6$

if $d = 3$ then $T_i = 2,\ T_{i+1} = 5,\ T_{i+2} = 8$

if $d = -6$ then $T_i = 11\quad T_{i+1} = 5\quad T_{i+2} = -1$

11 The first, third and ninth terms of an arithmetic series are also the first three terms of a geometric series. Find the common ratio of the geometric series.

T_1, T_3, T_9 are such that $\dfrac{T_9}{T_3} = \dfrac{T_3}{T_1} = r$

$T_3 = T_1 + 2d$ and $T_9 = T_1 + 8d$

Further $T_9 = r T_3 = r^2 T_1$

$$\frac{T_1 + 8d}{T_1 + 2d} = \frac{T_1 + 2d}{T_1} \Rightarrow \cancel{T_1^2} + 8d\,T_1 = \cancel{T_1^2} + 4d\,T_1 + 4d^2$$

$$\Rightarrow 4d\,T_1 = 4d^2$$

so $T_1 = d$.

$$\therefore \frac{T_1 + 2d}{T_1} = \frac{d + 2d}{d} = 3$$

18 This table gives the future value of an annuity of $1 at the given interest rate for the given period.

Future value interest factors					
$1	**Interest rate per period**				
N	**1%**	**2%**	**3%**	**4%**	**5%**
1	1.0000	1.0000	1.0000	1.0000	1.0000
2	2.0100	2.0200	2.0300	2.0400	2.0500
3	3.0301	3.0604	3.0909	3.1216	3.1525
4	4.0604	4.1216	4.1836	4.2465	4.3101
5	5.0101	5.2040	5.3091	5.4163	5.5256
6	6.1520	6.3081	6.4684	6.6330	6.8019

Use the table of future value interest factors to find:

(a) the future value of an annuity of $4500 per year for 6 years at 4% per annum

(b) the future value of an annuity of $700 per year for 5 years at 2% per annum.

a) $4500 \times 6.6330 = 29{,}848.50$

b) $700 \times 5.2040 = 3{,}642.80$

FINANCIAL MATHEMATICS - CHAPTER REVIEW

20 The table gives the present value interest factors for an annuity of $1 per period, for various interest rates, r, and number of periods, N.

Present value interest factors					
r / N	Interest rate per period (as a decimal)				
	0.0025	**0.005**	**0.0075**	**0.008**	**0.009**
71	64.981 40	59.641 21	54.892 93	54.007 54	52.296 57
72	65.816 86	60.339 51	55.476 85	54.570 97	52.821 18
73	66.650 23	61.034 34	56.056 43	55.129 93	53.341 11
74	67.481 53	61.725 71	56.631 69	55.684 46	53.856 41
75	68.310 75	62.413 65	57.202 67	56.234 58	54.367 10
76	69.137 91	63.098 15	57.769 40	56.780 34	54.873 24

(a) Randall plans to invest $200 each month for 74 months. Her investment will earn interest at a rate of 0.005 (as a decimal) per month. Use the information in the table to calculate the present value of this annuity.

(b) Rosa uses the same table to calculate the loan repayments for her car loan. Her loan is for $19 000 and will be repaid in equal monthly repayments over 5 years and 11 months. The interest rate on her loan is 9% per annum. Calculate the amount of each monthly repayment, rounded to the next dollar.

a) $PV = 200 \times 61.72571 = 12,345.142$

b) 5 year 11 maths is 71 months

9% p.a is $\frac{9}{12} = 0.75\%$ p.m $= 0.0075$ p.m.

So $\frac{19,000}{54.89293} = \$346.13$ per math.

21 Malcolm deposits \$3000 in an investment account that is paying a monthly interest rate of 0.4%, with the interest compounded monthly. Calculate the value of the investment after 24 months.

$$P = 3{,}000\,(1 + 0.004)^{24} = 3{,}301.64$$

22 Tanya and Alan plan to have \$20 000 in an investment account in 5 years time to pay for a holiday. The interest rate for the account will be fixed at 3.6% per annum, compounded monthly. How much do they need to deposit into the account to achieve this goal? Round your answer to the next dollar.

$$20{,}000 = A\left(1 + \frac{3.6}{12 \times 100}\right)^{5 \times 12}$$

$$\text{So}\quad A = \frac{20{,}000}{\left(1 + \frac{3.6}{12 \times 100}\right)^{60}} = 16{,}709.91$$

$$\simeq \$16{,}710$$

23 Sanjay borrows \$5000 at an interest rate of 1.5% per month and pays it off in equal monthly instalments. What should the instalment be in order to pay off the loan at the end of 3 years?

3 years is $3 \times 12 = 36$ months

Debt owed after the 1st payment $= D_1 = 5{,}000 \times 1.015 - P$

——————— 2nd payment $= D_2 = [5{,}000 \times 1.015 - P] \times 1.015 - P$

$$D_2 = 5{,}000 \times 1.015^2 - P(1 + 1.015)$$

Debt owed after the 3rd payment =

$$D_3 = [5{,}000 \times 1.015^2 - P(1 + 1.015)] \times 1.015 - P$$

$$D_3 = 5{,}000 \times 1.015^3 - P(1 + 1.015 + 1.015^2)$$

So Debt owed after the n payment

$$D_n = 5{,}000 \times 1.015^n - P(\underbrace{1 + 1.015 + \dots + 1.015^{n-1}}_{= \frac{1.015^n - 1}{1.015 - 1}})$$

$$D_n = 5{,}000 \times 1.015^n - \frac{P(1.015^n - 1)}{0.015}$$

So for D_{36} to be zero, when $n = 36$, we must have

$$0 = 5{,}000 \times 1.015^{36} - \frac{P(1.015^{36} - 1)}{0.015}$$

$$P = \frac{(5{,}000 \times 1.015^{36}) \times 0.015}{1.015^{36} - 1} = \$180.76 \text{ per month}$$

24 (a) At the beginning of each year, $100 is placed in an investment fund. Calculate the accumulated value at the end of 12 years if the interest is 6% p.a. compounded yearly.

(b) If the $100 due at the beginning of the fifth year was not placed in the fund, what would be the accumulated value at the end of 12 years?

a) $S = 100 \times 1.06^{12} + 100 \times 1.06^{11} + \cdots + 100 \times 1.06$

$S = 100 \times 1.06 \, [\underbrace{1 + 1.06 + \cdots + 1.06^{11}}_{= \frac{1.06^{12}-1}{1.06-1}}]$

$S = \dfrac{100 \times 1.06 \times (1.06^{12}-1)}{0.06} = \$1,788.21$

b) $S' = 100 \times 1.06^{12} + 100 \times 1.06^{11} + \cdots + 100 \times 1.06^{6} + 100 \times 1.06^{4} + \cdots$
$\cdots + 100 \times 1.06^{1}$

$S' = 100 \times 1.06^{6} \left[1 + 1.06 + 1.06^{2} + \cdots + 1.06^{6}\right] +$

$100 \times 1.06 \left[1 + \cdots + 1.06^{3}\right]$

$S' = 100 \times 1.06^{6} \left[\dfrac{1.06^{7}-1}{0.06}\right] + 100 \times 1.06 \left[\dfrac{1.06^{4}-1}{0.06}\right]$

$S' = 1,190.68 + 463.71$

$S' = 1,654.39$

<u>Note</u>: the book says $1,628.83 but I don't understand how they got that

25 A family borrows \$300 000 from a bank. Interest is charged at 6% p.a.

(a) How much must be repaid each year, rounded to the next dollar, if the loan is to be repaid over 30 years?

(b) How much of the loan will remain after the 18th payment?

(c) If repayments are made at \$40 000 per year, how long will it take to repay the loan? What will be the last repayment?

a) Debt after 1st payment is $D_1 = \$300,000 \times 1.06 - P$

2nd payment is $D_2 = [300,000 \times 1.06 - P] \times 1.06 - P$

$$D_2 = 300,000 \times 1.06^2 - P[1 + 1.06]$$

$$D_{30} = [300,000 \times 1.06^{30} - P(1 + 1.06 + \dots + 1.06^{29})$$

$$D_{30} = 300,000 \times 1.06^{30} - P\left[\frac{1.06^{30} - 1}{1.06 - 1}\right]$$

So as $D_{30} = 0$, $P = \dfrac{300,000 \times 1.06^{30} \times 0.06}{1.06^{30} - 1}$

$$P = \$21,795$$

b) $D_{18} = 300,000 \times 1.06^{18} - 21,795 \times \left[\dfrac{1.06^{18} - 1}{0.06}\right]$

$$D_{18} = \$182,713.05$$

c) $D_n = 300,000 \times 1.06^n - 40,000\left[\dfrac{1.06^n - 1}{0.06}\right] = 0$

So $30 \times 0.06 \times 1.06^n - 4 \times 1.06^n = -4$

So $1.06^n [1.8 - 4] = -4 \iff 1.06^n = \dfrac{20}{11}$

$n \times \ln 1.06 = \ln(20/11)$ $n = 10.26$ so 11

$D_{10} = 300,000 \times 1.06^{10} - 40,000\left[\dfrac{1.06^{10} - 1}{0.06}\right] = 10,022.51$ is the last payment

NOTE: the book says 10,623.86