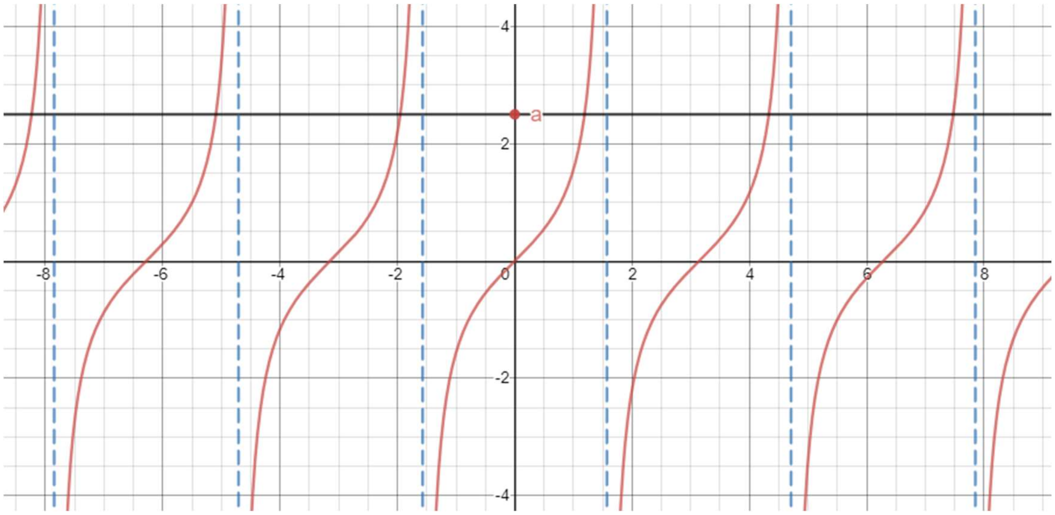
OVERVIEW OF TRIGONOMETRIC EQUATIONS

1. Equations of the form $\boldsymbol{a = \tan\theta}$ ("a" being a constant)



The obvious solution is $\theta = \tan^{-1} a$

But there are other solutions as *tangent* is periodic of period π

The general solution is $$\boldsymbol{\theta = (\tan^{-1} a) + n\pi}$$ where n is an integer

Example 18

Find all the angles for which $\tan\theta = 1$, where θ is in radians.

Solution

$\tan\theta = 1$

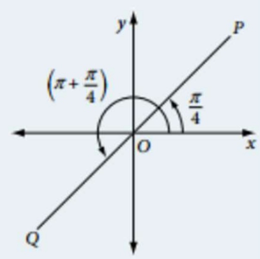
i.e. $\tan\theta = \tan\frac{\pi}{4}, \tan\left(\pi + \frac{\pi}{4}\right), \tan\left(2\pi + \frac{\pi}{4}\right), \ldots$

In the diagram, OP defines the angles: $\frac{\pi}{4}, 2\pi + \frac{\pi}{4}, 4\pi + \frac{\pi}{4}, \ldots$

and OQ defines the angles: $\pi + \frac{\pi}{4}, 3\pi + \frac{\pi}{4}, 5\pi + \frac{\pi}{4}, \ldots$

These results can be summarised as: $\theta = n\pi + \frac{\pi}{4}$ where n is any integer

or: $\theta = n \times 180° + 45°$ (in degrees)



Example 19

Solve $\tan x = 3\cot x$ for $-\pi \le x \le \pi$.

Solution

$\tan x = 3\cot x$

$\tan x = \frac{3}{\tan x}$

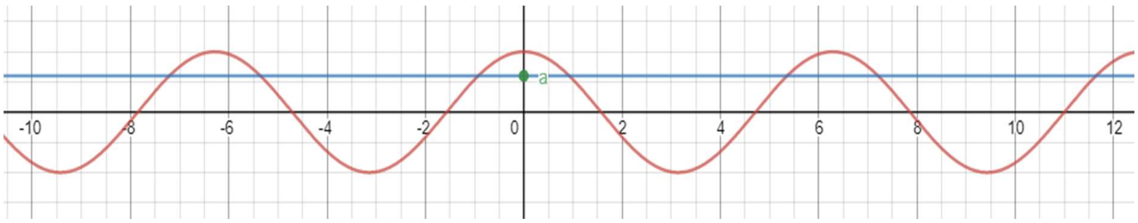
$\tan^2 x = 3$

$\tan x = \pm\sqrt{3}$

Solution is in all four quadrants: $x = -\pi + \frac{\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \pi - \frac{\pi}{3}$

$x = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}$

2. Equations of the form $\boldsymbol{a = \cos\theta}$ ("a" being a constant)



If $a > 1$ or $a < -1$, there are no solutions. Otherwise, the obvious solutions are $\begin{cases} \theta = \cos^{-1} a \\ \theta = -\cos^{-1} a \end{cases}$

But there are other solutions as *cosine* is periodic of period 2π

Therefore all solutions are $\begin{cases} \theta = (\cos^{-1} a) + n \times 2\pi \\ \theta = (-\cos^{-1} a) + n \times 2\pi \end{cases}$ where n is an integer

or simply written $\begin{cases} \theta = \cos^{-1} a + 2n\pi \\ \theta = -\cos^{-1} a + 2n\pi \end{cases}$

which summarises as $\boldsymbol{\theta = \pm\cos^{-1} a + 2n\pi}$ where n is an integer

Example 16

Find all angles θ for which $\cos\theta = \frac{1}{2}$.

Solution

$\cos\theta = \frac{1}{2}$

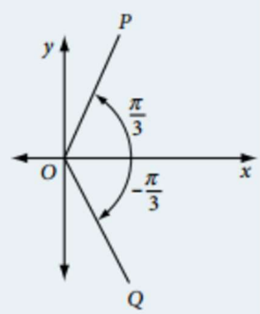
i.e. $\cos\theta = \cos\frac{\pi}{3}, \cos\left(2\pi - \frac{\pi}{3}\right), \cos\left(2\pi + \frac{\pi}{3}\right), \ldots$

In the diagram, OP defines the angles: $\frac{\pi}{3}, 2\pi + \frac{\pi}{3}, 4\pi + \frac{\pi}{3}, \ldots$

and OQ defines the angles: $-\frac{\pi}{3}, 2\pi - \frac{\pi}{3}, 4\pi - \frac{\pi}{3}, \ldots$

These results can be summarised as: $\theta = 2n\pi \pm \frac{\pi}{3}$ where n is any integer

or: $\theta = n \times 360° \pm 60°$ (in degrees)



Example 17

Solve $2\cos(3x + 30°) + \sqrt{3} = 0$ for $0° \le x \le 360°$.

Solution

$2\cos(3x + 30°) = -\sqrt{3}$

$\cos(3x + 30°) = -\frac{\sqrt{3}}{2}$

$\cos\theta < 0$ in second and third quadrants: $\cos(3x + 30°) = \cos 150°, \cos 210°, \ldots$

As $0° \le x \le 360°$, thus $0° \le 3x \le 3 \times 360°$, so two more revolutions are needed.

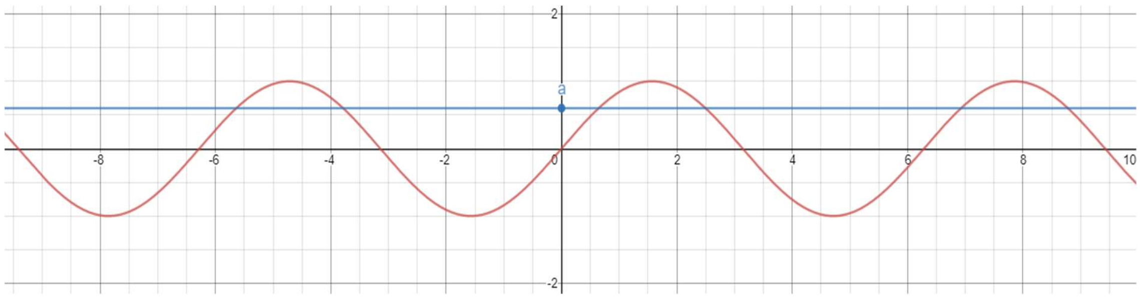
$3x + 30° = 150°, 210°, 510°, 570°, 870°, 930°$

$3x = 120°, 180°, 480°, 540°, 840°, 900°$

$x = 40°, 60°, 160°, 180°, 280°, 300°$

OVERVIEW OF TRIGONOMETRIC EQUATIONS

3. Equations of the form $\boldsymbol{a = \sin\theta}$ ("a" being a constant)



If $a > 1$ or $a < -1$, there are no solutions. Otherwise, the obvious solutions are $\begin{cases} \theta = \sin^{-1} a \\ \theta = \pi - \sin^{-1} a \end{cases}$

But there are other solutions as *sine* is periodic of period 2π

Therefore all solutions are $\begin{cases} \theta = (\sin^{-1} a) + k \times 2\pi \\ \theta = (\pi - \sin^{-1} a) + k \times 2\pi \end{cases}$ where k is an integer

or simply written $\begin{cases} \theta = \sin^{-1} a + 2k\pi \\ \theta = \pi - \sin^{-1} a + 2k\pi \end{cases}$

or $\begin{cases} \theta = \sin^{-1} a + 2k\pi \\ \theta = -\sin^{-1} a + (2k+1)\pi \end{cases}$

which can also be written as $\begin{cases} \theta = \mathbf{1} \times \sin^{-1} a + 2k\pi \\ \theta = (\mathbf{-1}) \times \sin^{-1} a + (2k+1)\pi \end{cases}$

Now noting that $\mathbf{1} = (-1)^{2k}$ and that $(\mathbf{-1}) = (-1)^{2k+1}$

the equations transform as $\begin{cases} \theta = (-1)^{2k} \times \sin^{-1} a + 2k\pi \\ \theta = (-1)^{2k+1} \times \sin^{-1} a + (2k+1)\pi \end{cases}$

or $\begin{cases} \theta = (-1)^{n} \times \sin^{-1} a + n\pi & \text{with } n\text{" an } \textbf{even} \text{ integer} \\ \theta = (-1)^{n} \times \sin^{-1} a + n\pi & \text{with "}n\text{" an } \textbf{odd} \text{ integer} \end{cases}$

which summarises as $\boldsymbol{\theta = (-1)^n \times \sin^{-1} a + n\pi}$ where n is an integer

OVERVIEW OF TRIGONOMETRIC EQUATIONS

Example 14

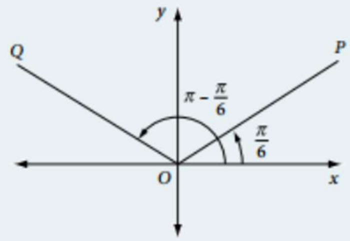
Find all values of θ for which $\sin\theta = \frac{1}{2}$.

Solution

$$\sin\theta = \frac{1}{2}$$

$$\therefore\ \sin\theta = \sin\frac{\pi}{6},\ \sin\left(\pi - \frac{\pi}{6}\right),\ \sin\left(2\pi + \frac{\pi}{6}\right), \ldots$$

Consider a coordinate diagram.



$$\angle XOP = \frac{\pi}{6}$$

$$\angle XOQ = \pi - \frac{\pi}{6}$$

The ray OP defines an infinite number of angles in the first quadrant. If you rotate OP about the origin (either clockwise or anticlockwise), then during each revolution it is along the original ray OP once.
Each full rotation increases the angle by 2π, so you find that OP is the terminal ray defining the angles:

- $\frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 4\pi + \frac{\pi}{6}, \ldots$ for anticlockwise rotation
- $-2\pi + \frac{\pi}{6}, -4\pi + \frac{\pi}{6}, -6\pi + \frac{\pi}{6}, \ldots$ for clockwise rotation.

This result can be summarised as: $n\pi + \frac{\pi}{6}$ where $n = 0, \pm 2, \pm 4, \ldots$ [1]

or: $n \times 180° + 30°$ (in degrees)

Similarly, the terminal ray OQ defines an infinite number of angles:

$\pi - \frac{\pi}{6}, 3\pi - \frac{\pi}{6}, 5\pi - \frac{\pi}{6}, \ldots$ for anticlockwise rotation

$-\pi - \frac{\pi}{6}, -3\pi - \frac{\pi}{6}, -5\pi - \frac{\pi}{6}, \ldots$ for clockwise rotation.

This result can be summarised as: $n\pi - \frac{\pi}{6}$ where $n = \pm 1, \pm 3, \pm 5, \ldots$ [2]

or: $n \times 180° - 30°$ (in degrees)

Statements [1] and [2] can be written together as:

$$\theta = n\pi + (-1)^n \frac{\pi}{6}$$

or: $\theta = n \times 180° + (-1)^n \times 30°$ (in degrees)

Note: $(-1)^n$ is 1 when n is zero or even, and is -1 when n is odd.

Example 15

Solve $\sin\left(\theta + \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$ for $0 \le \theta \le 2\pi$.

Solution

$$\sin\left(\theta + \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$\sin\theta < 0$ in third and fourth quadrants: $\sin\left(\theta + \frac{\pi}{4}\right) = \sin\frac{5\pi}{4}, \sin\frac{7\pi}{4}$

$$\theta + \frac{\pi}{4} = \frac{5\pi}{4}, \frac{7\pi}{4}$$

$$\theta = \pi, \frac{3\pi}{2}$$

These 3 formulas are NOT in the Maths Extension formula sheet.