INTEGRATION BY PARTS

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2 Find: **(a)** $\int x\sqrt{4-x}\,dx$ **(b)** $\int x\tan^{-1}x\,dx$

a) $\int u\,dv = uv - \int v\,du$ let $\frac{dv(x)}{dx} = \sqrt{4-x}$ and $u(x) = x$

So $v(x) = (-1)\frac{(4-x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}$ $u'(x) = 1$

$$\int x\sqrt{4-x}\,dx = -\frac{2x}{3}(4-x)^{3/2} - \int -\frac{2}{3}(4-x)^{3/2}\,dx$$

$$= -\frac{2x}{3}(4-x)^{3/2} + \frac{2}{3}\int (4-x)^{3/2}\,dx$$

$$= -\frac{2x}{3}(4-x)^{3/2} + \frac{2}{3}\frac{(4-x)^{3/2+1}}{\left(\frac{3}{2}+1\right)} \times (-1) + C$$

$$= -\frac{2x}{3}(4-x)^{3/2} - \frac{4}{15}(4-x)^{5/2} + C$$

b) $\int x\tan^{-1}x\,dx$ let $u(x) = \tan^{-1}x$ and $v'(x) = x$

$v(x) = \frac{x^2}{2}$

So $u'(x) = \frac{1}{1+x^2}$

$$\int x\tan^{-1}x\,dx = \frac{x^2}{2}\tan^{-1}x - \int \frac{x^2}{2} \times \frac{1}{1+x^2}\,dx$$

$$\int x\tan^{-1}x\,dx = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$$

$$\int x\tan^{-1}x\,dx = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\left[\int \frac{x^2+1-1}{1+x^2}\,dx\right]$$

$$\int x\tan^{-1}x\,dx = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx$$

$$\int x\tan^{-1}x\,dx = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x + C$$

$$= (\tan^{-1}x)\left(\frac{x^2+1}{2}\right) - x/2 + C$$

INTEGRATION BY PARTS

2 Find: **(d)** $\int x \sin 2x \, dx$ **(f)** $\int \sin^{-1} 2x \, dx$

d) $u(x) = x$ $\quad v(x) = -\dfrac{\cos 2x}{2}$

$u'(x) = 1$ $\quad v'(x) = \sin 2x$

$$\int x \sin 2x \, dx = x\left(\frac{-\cos 2x}{2}\right) - \int \left(\frac{-\cos 2x}{2}\right) dx$$

$$= -\frac{x \cos 2x}{2} + \frac{1}{2}\int \cos 2x \, dx$$

$$= -\frac{x \cos 2x}{2} + \frac{1}{2}\,\frac{\sin 2x}{2} + C$$

$$= -\frac{x \cos 2x}{2} + \frac{\sin 2x}{4} + C$$

f) $u(x) = \sin^{-1} 2x$ $\quad v(x) = x$

$u'(x) = \dfrac{1}{\sqrt{1-(2x)^2}} \times 2$ $\quad v'(x) = 1$

$$\int \sin^{-1} 2x \, dx = x \sin^{-1} 2x - \int \frac{2x}{\sqrt{1-4x^2}} dx$$

For the 2nd term, let $u = 1-4x^2$ so $\dfrac{du}{dx} = -8x$

so $du = -8x \, dx = 4 \times (-2x \, dx)$

$$\int \sin^{-1} 2x \, dx = x \sin^{-1} 2x + \frac{1}{4}\int \frac{du}{u^{1/2}}$$

$$= x \sin^{-1} 2x + \frac{1}{4}\,\frac{u^{-1/2+1}}{\left(-\frac{1}{2}+1\right)} + C$$

$$= x \sin^{-1} 2x + \frac{1}{2} u^{1/2} + C$$

$$= x \sin^{-1} 2x + \frac{1}{2}\sqrt{1-4x^2} + C$$

INTEGRATION BY PARTS

Logarithmic
Inverse trig
Algebraic
Trigonom.
Exponential

3 Find: **(d)** $\int x^2 \cos x \, dx$ **(e)** $\int x e^{-x} \, dx$

d) $u(x) = x^2$ $v(x) = \sin x$

$u'(x) = 2x$ $v'(x) = \cos x$

$\int x^2 \cos x \, dx = x^2 \sin x - \int 2x \sin x \, dx$

$\_\_\_ = x^2 \sin x - 2 \int x \sin x \, dx$ ← another integration by parts.

$u(x) = x$ $v(x) = -\cos x$

$u'(x) = 1$ $v'(x) = \sin x$

$\_\_\_ = x^2 \sin x - 2\left[-x\cos x - \int -\cos x \, dx\right]$

$\_\_\_ = x^2 \sin x + 2x\cos x - 2\int \cos x \, dx + C$

$\_\_\_ = x^2 \sin x + 2x\cos x - 2\sin x + C$

e) $u(x) = x$ $v(x) = -e^{-x}$

$u'(x) = 1$ $v'(x) = e^{-x}$

$\int x e^{-x} \, dx = -x e^{-x} - \int 1 \times (-e^{-x}) \, dx$

$\_\_\_ = -x e^{-x} + \int e^{-x} \, dx$

$\_\_\_ = -x e^{-x} - e^{-x} + C$

$\_\_\_ = -e^{-x}(x+1) + C$

INTEGRATION BY PARTS

4 Find: **(a)** $\int e^{-x}\sin x\,dx$ **(d)** $\int \sin(\log_e x)\,dx = I$

a) $u(x) = \sin x \qquad v(x) = -e^{-x}$

$u'(x) = \cos x \qquad v'(x) = e^{-x}$

$\int e^{-x}\sin x\,dx = -e^{-x}\sin x - \int -e^{-x}\cos x\,dx$

$= -e^{-x}\sin x + \int e^{-x}\cos x\,dx$ — another integration by part

$u(x) = \cos x \qquad v(x) = -e^{-x}$

$u'(x) = -\sin x \qquad v'(x) = e^{-x}$

So $\int e^{-x}\cos x\,dx = -e^{-x}\cos x - \int e^{-x}\sin x\,dx$

So $\int e^{-x}\sin x\,dx = -e^{-x}\sin x - e^{-x}\cos x - \int e^{-x}\sin x\,dx$

$\therefore\ 2\int e^{-x}\sin x\,dx = -e^{-x}(\sin x + \cos x) + C \quad \therefore \int e^{-x}\sin x\,dx = \frac{-e^{-x}(\sin x+\cos x)}{2} + C$

d) $u(x) = \sin(\ln x) \qquad v(x) = x$ ⚠ LIATE doesn't work here!

$u'(x) = \frac{\cos(\ln x)}{x} \qquad v'(x) = 1$

$I = x\sin(\ln x) - \int \frac{\cos(\ln x)}{x} \times x\,dx = x\sin(\ln x) - \int \cos(\ln x)\,dx.$ another integration by parts

$u(x) = \cos(\ln x) \qquad v(x) = x$

$u'(x) = -\frac{\sin(\ln x)}{x} \qquad v'(x) = 1$

$I = x\sin(\ln x) - \left[x\cos(\ln x) - \int -\frac{\sin(\ln x)}{x} \times x\,dx\right]$

$I = x\sin(\ln x) - x\cos\ln x - \underbrace{\int \sin(\ln x)\,dx}_{= I}$

So $2I = x\left[\sin(\ln x) - \cos(\ln x)\right] + C$

$I = \int \sin(\ln x)\,dx = \frac{x}{2}\left[\sin(\ln x) - \cos(\ln x)\right] + C$

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INTEGRATION BY PARTS

5 **Evaluate:** **(a)** $\int_0^{\frac{\pi}{2}} x\cos x\,dx$ **(e)** $\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \sin^{-1} x\,dx$

a) $u(x) = x$ $\quad v(x) = \sin x$

$u'(x) = 1$ $\quad v'(x) = \cos x$

$I = \int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x - (-\cos x) + C$

So $\int x\cos x\,dx = x\sin x + \cos x + C$

$\therefore \int_0^{\pi/2} x\cos x\,dx = \left[x\sin x + \cos x\right]_0^{\pi/2} = \left[\frac{\pi}{2} + 0\right] - \left[0 + 1\right] = \frac{\pi}{2} - 1$

b) $u(x) = \sin^{-1} x$ $\quad v(x) = x$

$u'(x) = \frac{1}{\sqrt{1-x^2}}$ $\quad v'(x) = 1$

$\int \sin^{-1} x\,dx = x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\,dx$

Change of variable $u = 1 - x^2$ so $\frac{du}{dx} = -2x$

$du = -2x\,dx \iff -x\,dx = \frac{du}{2}$

$\int \sin^{-1} x\,dx = x\sin^{-1} x + \frac{1}{2}\int \frac{du}{u^{1/2}} = x\sin^{-1} x + \frac{1}{2}\frac{u^{-\frac{1}{2}+1}}{\left(-\frac{1}{2}+1\right)} + C$

$\int \sin^{-1} x\,dx = x\sin^{-1} x + \sqrt{u} + C$

$\int \sin^{-1} x\,dx = x\sin^{-1} x + \sqrt{1-x^2} + C$

$\therefore \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \sin^{-1} x\,dx = \left[x\sin^{-1} x + \sqrt{1-x^2}\right]_{-1/\sqrt{2}}^{1/\sqrt{2}}$

$= \left[\frac{1}{\sqrt{2}} \times \frac{\pi}{4} + \sqrt{\frac{1}{2}}\right] - \left[\frac{-1}{\sqrt{2}} \times \left(\frac{-\pi}{4}\right) + \sqrt{\frac{1}{2}}\right] = 0$

NOTE: we could / should have concluded that straight away as $f(x) = \sin^{-1} x$ is an odd function, $\therefore \int_{-a}^{a} f(x) = 0$ ☹

INTEGRATION BY PARTS

6 Evaluate: (b) $\int_0^{\frac{\pi}{2}} e^x \sin x\,dx$ (h) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} x\sec^2 x\,dx$

b) $u(x) = \sin x$ $\quad v(x) = e^x$

$u'(x) = \cos x$ $\quad v'(x) = e^x$

$$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx$$ another integration by parts.

$u(x) = \cos x$ $\quad v(x) = e^x$

$u'(x) = -\sin x$ $\quad v'(x) = e^x$

$$\therefore \int e^x \sin x\,dx = e^x \sin x - \left[e^x \cos x - \int e^x(-\sin x)\,dx\right]$$

$$\therefore 2\int e^x \sin x\,dx = e^x(\sin x - \cos x)$$

$$\therefore \int e^x \sin x\,dx = \frac{e^x}{2}(\sin x - \cos x)$$

$$\therefore \int_0^{\pi/2} e^x \sin x\,dx = \left[\frac{e^{\pi/2}}{2} \times 1\right] - \left[\frac{1}{2}(0-1)\right] = \frac{e^{\pi/2}}{2} + \frac{1}{2}$$

h) $u(x) = x$ $\quad v(x) = \tan x$

$u'(x) = 1$ $\quad v'(x) = \sec^2 x$

$$\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx = x\tan x - \int \frac{\sin x}{\cos x}\,dx$$

$$= x\tan x + \int \frac{(\cos x)'}{\cos x}\,dx$$

$$= x\tan x + \ln|\cos x| + C$$

$$\therefore \int_{\pi/6}^{\pi/3} x\sec^2 x\,dx = \Big[x\tan x + \ln|\cos x|\Big]_{\pi/6}^{\pi/3}$$

$$= \left[\frac{\pi}{3} \times \sqrt{3} + \ln\left(\frac{1}{2}\right)\right] - \left[\frac{\pi}{6} \times \frac{1}{\sqrt{3}} + \ln\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$= \frac{\pi}{\sqrt{3}}\left(1 - \frac{1}{6}\right) + \ln\left(\frac{1/2}{\sqrt{3}/2}\right) = \frac{5\pi}{6\sqrt{3}} - \ln\sqrt{3} = \frac{5\sqrt{3}\pi}{18} - \frac{1}{2}\ln 3$$

7 $\int_{-\pi}^{\pi} x^2 \sin x \, dx = \ldots$

A $2\pi^2 - 2$ $\boxed{\textbf{B}}$ 0 **C** 2 **D** $2\pi^2$

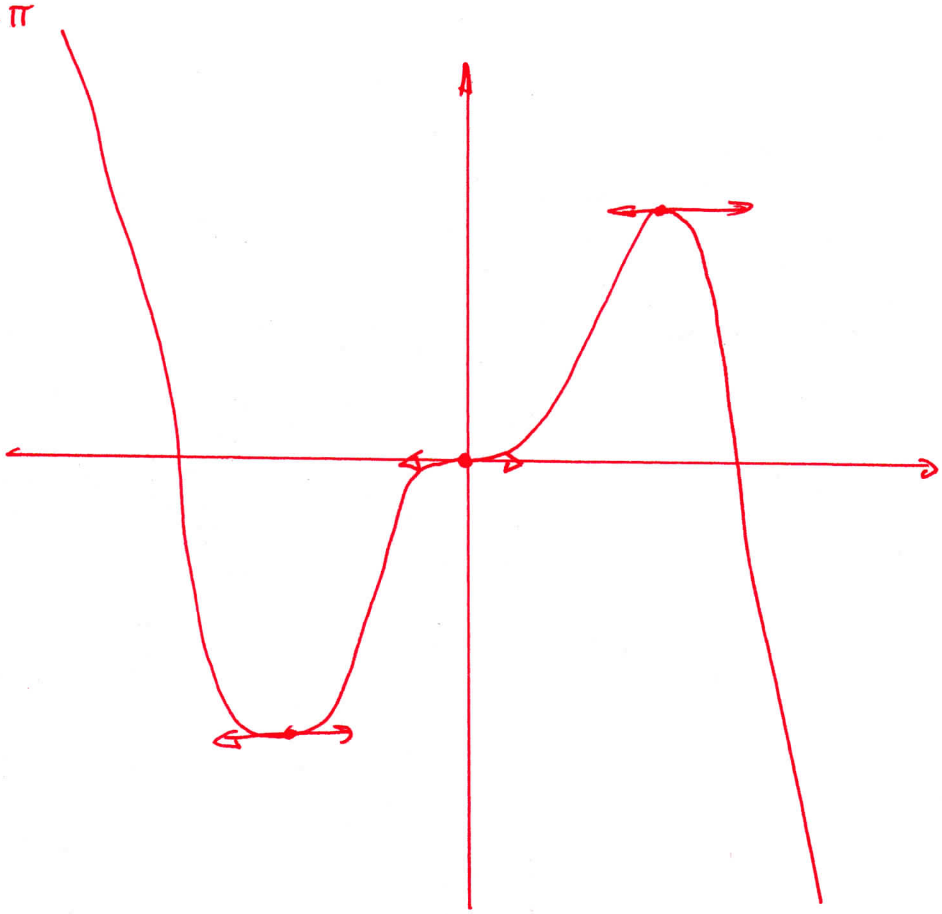
Let $f(x) = x^2 \sin x$

$f(-x) = x^2 \sin(-x) = -x^2 \sin x = -f(x)$

$\therefore$ f is an odd function

$\therefore$ $\forall a \in \mathbb{R}$ $\int_{-a}^{a} f(x)\, dx = 0$

$\therefore$ $\int_{-\pi}^{\pi} x^2 \sin x \, dx = 0$



INTEGRATION BY PARTS

8 Find the area of the region bounded by the curve $y = \log_e x$ $(x > 0)$, the x-axis and the line $x = a$ $(a > 1)$.

We are looking for $\int_1^a \ln x \, dx$

$u(x) = \ln x \qquad v(x) = x$

$u'(x) = \frac{1}{x} \qquad v'(x) = 1$

$$\int_1^a \ln x \, dx = \left[x \ln x\right]_1^a - \int_1^a x \times \frac{1}{x} \, dx$$

$$\text{———} = a \ln a - \int_1^a dx$$

$$\text{———} = a \ln a - \left[x\right]_1^a$$

$$\text{———} = a \ln a - a + 1$$