ABSOLUTE VALUE FUNCTIONS

Absolute value

The absolute value (also called 'modulus') of a real number x is written $|x|$. It is the non-negative number that defines the magnitude of the given number.
Thus $|3| = 3, |-3| = 3$ and $|0| = 0$.

This means:

$$|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$$

This is identical to $\sqrt{x^2}$, so it leads to another definition of absolute value: $|x| = \sqrt{x^2}$.

Important results

1 $|xy| = |x| \times |y|$

2 $|x + y| \leq |x| + |y|$ (the 'triangle inequality')
and $|x + y| = |x| + |y|$ if and only if x and y are either zero or have the same sign.

Example 9

On a number line, show the values of x for which:

(a) $|x| > 1$ **(b)** $|x| \leq 2$

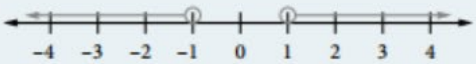
Solution

(a) $|x| > 1$

$x > 1$ or $-x > 1$

$x > 1$ or $x < -1$

$x < -1$ or $x > 1$

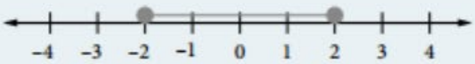


(b) $|x| \leq 2$

$x \leq 2$ or $-x \leq 2$

$x \leq 2$ or $x \geq -2$

$-2 \leq x \leq 2$



When the circle is filled in, the point is included, as it is in part **(b)**.

Example 10

Solve for x:

(a) $|2x-1|=3$ **(b)** $|3x+2|=1$ **(c)** $|2x-1|\geq 3$ **(d)** $|3x+2|<1$

Solution

(a) $|2x-1|=3$

$$2x-1=3 \quad \text{or} \quad -(2x-1)=3$$
$$2x-1=3 \quad \text{or} \quad 2x-1=-3$$
$$2x=4 \quad \text{or} \quad 2x=-2$$
$$x=2 \quad \text{or} \quad x=-1$$

(b) $|3x+2|=1$

$$3x+2=1 \quad \text{or} \quad 3x+2=-1$$
$$3x=-1 \quad \text{or} \quad 3x=-3$$
$$x=-\frac{1}{3} \quad \text{or} \quad x=-1$$

(c) $|2x-1|\geq 3$

$$2x-1\geq 3 \quad \text{or} \quad -(2x-1)\geq 3$$

(d) $|3x+2|<1$

$$3x+2<1 \quad \text{or} \quad -(3x+2)<1$$

Multiplying both sides of an inequality by −1 reverses the direction of the inequality, so:

(c)

$$2x-1\geq 3 \quad \text{or} \quad 2x-1\leq -3$$
$$2x\geq 4 \quad \text{or} \quad 2x\leq -2$$
$$x\geq 2 \quad \text{or} \quad x\leq -1$$

(d)

$$3x+2<1 \quad \text{or} \quad 3x+2>-1$$
$$3x<-1 \quad \text{or} \quad 3x>-3$$
$$x<-\frac{1}{3} \quad \text{or} \quad x>-1$$
$$-1<x<-\frac{1}{3}$$

In **(b)** above, the first line of working has been left out. When you are confident solving absolute value equations, you can do this too. However, beware that skipping the first line of working in problems like parts **(c)** and **(d)** could easily lead to wrong inequality signs.

ABSOLUTE VALUE FUNCTIONS

You can now consider $f(x) = |x|$, which is called the absolute value function (or the numerical value function).

From our earlier definition of absolute value, you have: $f(x) = |x| = \begin{cases} x & x \in (0, \infty) \\ -x & x \in (-\infty, 0) \end{cases}$

Or, the alternative definition: $f(x) = |x| = \sqrt{x^2}$ for all real x

The domain of $f(x) = |x|$ is all real x. The range of $f(x) = |x|$ is non-negative real numbers (i.e. $f(x)$ is zero or a positive real number).

Example 13

Find the domain and range for each function. Sketch each function.

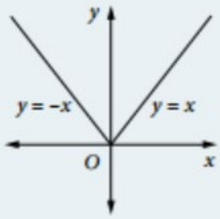
(a) $f(x) = \sqrt{x^2}$ **(b)** $f(x) = |2x - 1|$ **(c)** $f(x) = x + |x|$ **(d)** $f(x) = |x^2 - 4|$

Solution

(a) $f(x) = \sqrt{x^2} = |x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$

Domain: real x.

Range: $f(x)$ is zero or a positive real number.



(b) $f(x) = |2x - 1|$

$f(x) = 2x - 1$ for $2x - 1 \geq 0$

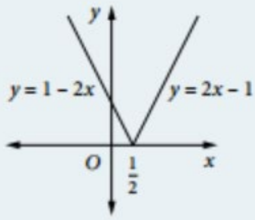
i.e. for $x \geq \frac{1}{2}$

$f(x) = -(2x - 1)$ for $2x - 1 < 0$

$f(x) = 1 - 2x$ for $x < \frac{1}{2}$

Domain: real x.

Range: $f(x)$ is zero or a positive real number.



(c) $f(x) = x + |x|$

$f(x) = x + x$ for $x \geq 0$

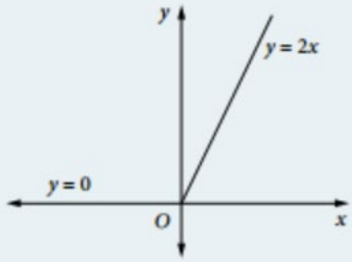
$f(x) = 2x$ for $x \geq 0$

$f(x) = x - x$ for $x < 0$

$f(x) = 0$ for $x < 0$

Domain: real x.

Range: $f(x)$ is zero or a positive real number.



(d) $f(x) = |x^2 - 4|$

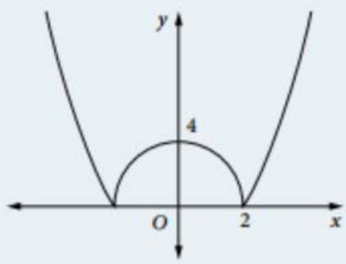
$f(x) = x^2 - 4$ for $x^2 - 4 \geq 0$

i.e. for $|x| \geq 2$

$f(x) = -(x^2 - 4)$ for $x^2 - 4 < 0$

$f(x) = 4 - x^2$ for $|x| < 2$

Domain: real x.



Range: $f(x)$ is zero or a positive real number.

This sketch can be obtained by first sketching the graph of $f(x) = x^2 - 4$. Take the part of that curve that is below the x-axis and reflect it above the x-axis.