INEQUALITIES

The relation $a > b$ is equivalent to the statement that $a - b$ is positive, i.e. $a - b > 0$. Similarly, you can interpret $a < b$ to mean either that $b > a$ or that $a - b$ is negative. The following properties are stated for inequalities with a '$>$' relation; similar properties exist for '$<$'.

Properties of inequalities

1 If $a > b$ then $a + x > b + x$ for all x.
You can add or subtract the same amount to both sides of an inequality.

2 If $a > b$ then $ax > bx$ for $x > 0$.
You can multiply (or divide) both sides of an inequality by a positive number.

3 If $a > b$ then $ax < bx$ for $x < 0$.
The inequality relation changes when you multiply (or divide) both sides of an inequality by a negative number.

4 If $a > b > 0$ then $\frac{1}{a} < \frac{1}{b}$.
When you take reciprocals of both sides of an inequality and both sides are positive, the inequality is reversed.

5 If $a > b > 0$ then $a^2 > b^2$.
Inequalities in which both sides are positive can be squared.

6 If $a > b$ and $b > c$ then $a > c$.
Inequalities of the same type can be linked together.

7 If $a > b > 0$ and $c > d > 0$ then $ac > bd$.
Inequalities of the same type involving positive numbers can be multiplied together.

Proof: $a > b$ and $c > 0$ $\quad \therefore ac > bc$
$c > d$ and $b > 0$ $\quad \therefore bc > bd$
$\therefore ac > bc > bd$ $\quad \therefore ac > bd$

8 If $a > b$ and $c > d$ then $a + c > b + d$.
Inequalities of the same type can be added together.

Proof: $a - b > 0$ and $c - d > 0$ $\quad \therefore a - b + c - d > 0$
$\therefore a + c - (b + d) > 0$
$\therefore a + c > b + d$

9 As a general rule, inequalities **cannot** be subtracted from one another, nor divided into each other.
For example:

Consider the pair of inequalities	$12 > 6$	[1]
and	$4 > 3$	[2]
[1] − [2] gives a true result:	$8 > 3$	
[1] ÷ [2] gives a true result:	$3 > 2$	
But now consider the inequalities	$12 > 10$	[3]
and	$100 > 3$	[4]
[3] − [4] gives a false result:	$-88 > 7$	
[3] ÷ [4] gives a false result:	$\frac{3}{25} > 3\frac{1}{3}$	

Techniques for proving inequalities

1 Use the properties of inequalities listed above.

2 Use proof by contradiction.

3 Use a known fact.
For example: $(a-b)^2 \geq 0$ for real a, b; or a variation of this, such as $(\sqrt{a}-\sqrt{b})^2 \geq 0$.
Another useful identity is $(a+b)^2 = (a-b)^2 + 4ab$, which enables the statement $(a+b)^2 \geq 4ab$ (as $(a-b)^2 \geq 0$ for real a, b).

4 Substitute different expressions into known inequalities.

5 If the inequality to be proved involves trigonometric, logarithmic or exponential terms, then a calculus-based approach is probably needed.

Example 13

If $a > -1$, show that $a^3 + 1 > a^2 + a$.

Solution

Technique: To prove $X > Y$, prove that $X - Y > 0$.

$$\begin{aligned} a^3 - a^2 - a + 1 &= a^2(a-1) - (a-1) \\ &= (a-1)(a^2-1) \\ &= (a-1)^2(a+1) \\ &> 0 \text{ as } a > -1 \text{, so } (a+1) \text{ is positive; also, } (a-1)^2 \text{ is positive} \end{aligned}$$

$\therefore a^3 + 1 > a^2 + a$.

Example 14

a, b, c and d are positive real numbers. Prove that:

(a) $a^2 + b^2 \geq 2ab$

(b) $a^2 + b^2 + c^2 \geq ab + bc + ca$

(c) $a^2 + b^2 + c^2 + d^2 \geq 2(ab + cd)$

(d) if $a + b + c = 1$, then $ab + bc + ca \leq \frac{1}{3}$, and state the condition for which equality is true

(e) $(a + b)^2 \geq 4ab$, and hence that $\left(\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}\right)^2 \geq 4$ for all real x except 0.

Solution

(a) **Method 1**

$(a - b)^2 \geq 0$ for real a, b

$\therefore\ a^2 - 2ab + b^2 \geq 0$

$a^2 + b^2 \geq 2ab$

Method 2—*Proof by contradiction*

Assume that $a^2 + b^2 < 2ab$ for all real a, b.

$\therefore\ a^2 - 2ab + b^2 < 0$

$\therefore\ (a - b)^2 < 0$ for real a, b

This contradicts the fact that the square of a real number is non-negative.

$\therefore$ Assumption is false, so: $a^2 + b^2 \geq 2ab$

(b)

$a^2 + b^2 \geq 2ab$ [1]

Similarly: $a^2 + c^2 \geq 2ac$ [2]

Similarly: $b^2 + c^2 \geq 2bc$ [3]

[1] + [2] + [3]: $2(a^2 + b^2 + c^2) \geq 2(ab + bc + ca)$

$a^2 + b^2 + c^2 \geq ab + bc + ca$

This shows how to extend from a known result involving two variables to a similar result involving three variables: use the known result to generate each of the three paired results, then add.

(c)

$a^2 + b^2 \geq 2ab$ [1]

Similarly: $c^2 + d^2 \geq 2cd$ [4]

[1] + [4]: $a^2 + b^2 + c^2 + d^2 \geq 2(ab + cd)$

This shows a method to extend from a known result involving two variables to a similar result involving four variables: use the known result to generate two suitable paired results, then add.

(d) The question now involves $a + b + c$, which suggests you examine $(a + b + c)^2$.

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$$

$$\geq 3(ab + bc + ca) \quad \text{using the result of part (b)}$$

But $a + b + c = 1$: $1^2 \geq 3(ab + bc + ca)$

$$\therefore \quad ab + bc + ca \leq \frac{1}{3}$$

Note that in the inequality $a^2 + b^2 \geq 2ab$ [1], equality occurs when $a = b$. Similarly, equality occurs in [2] when $a = c$. Hence the equality of this result in part **(d)** occurs when $a = b = c$. As $a + b + c = 1$, equality occurs when $a = b = c = \frac{1}{3}$.

(e) Using $(a + b)^2 = (a - b)^2 + 4ab$: you have $(a - b)^2 \geq 0$, so $(a + b)^2 \geq 4ab$.

Letting $a = \sqrt[3]{x}$ and $b = \frac{1}{\sqrt[3]{x}}$, you obtain: $\left(\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}\right)^2 \geq 4 \times \sqrt[3]{x} \times \frac{1}{\sqrt[3]{x}}$

$$\therefore \quad \left(\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}\right)^2 \geq 4$$

Example 15

a, b, c are positive real numbers. Prove that:

(a) $a+\frac{1}{a}\geq 2$ **(b)** $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)\geq 4$ **(c)** $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 9$

(d) Extend this to the general case. If $x_1, x_2, x_3, \ldots x_n$ are positive real numbers, prove that:

$$(x_1+x_2+x_3+\ldots+x_n)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\ldots+\frac{1}{x_n}\right)\geq n^2$$

Solution

(a) $\left(\sqrt{a}-\frac{1}{\sqrt{a}}\right)^2\geq 0$

$\therefore\ a-2+\frac{1}{a}\geq 0$

$a+\frac{1}{a}\geq 2$ This could also have been done very well by contradiction.

(b) $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)=1+\frac{a}{b}+\frac{b}{a}+1$

$=2+\left(A+\frac{1}{A}\right)$ with $A=\frac{a}{b}$

≥ 4 from part **(a)**

(c) $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=1+\frac{a}{b}+\frac{a}{c}+\frac{b}{a}+1+\frac{b}{c}+\frac{c}{a}+\frac{c}{b}+1$

$=3+\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{a}{c}+\frac{c}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)$

$\geq 3+2+2+2$ using part **(a)**

≥ 9

(d) $(x_1+x_2+x_3+\ldots+x_n)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\ldots+\frac{1}{x_n}\right)$

$=1+\frac{x_1}{x_2}+\frac{x_1}{x_3}+\ldots+\frac{x_1}{x_n}+\frac{x_2}{x_1}+1+\frac{x_2}{x_3}+\ldots+\frac{x_2}{x_n}+\ldots+\frac{x_n}{x_{n-1}}+1$

$=n\times 1+$ (every possible pairing of the form $\frac{x_i}{x_j}+\frac{x_j}{x_i}$ where $i\neq j$)

$\geq n+{}^nC_2\times 2$ (from the n different x terms, there are ${}^nC_2$ pairs)

$\geq n+\frac{1}{2}\times n(n-1)\times 2$

$\geq n^2$

$\therefore\ (x_1+x_2+x_3+\ldots+x_n)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\ldots+\frac{1}{x_n}\right)\geq n^2$

INEQUALITIES

Arithmetic and Geometric means

The arithmetic mean of a and b is $\frac{a+b}{2}$, the average of a and b.

The geometric mean of a and b is $\sqrt{ab}$. The numbers a, $\sqrt{ab}$, b form a geometric sequence.

Example 16

Prove that the arithmetic mean of two positive real numbers, a and b, is equal to or greater than the geometric mean, i.e. prove that $\frac{a+b}{2} \geq \sqrt{ab}$.

Solution

Consider the expression $\left(\frac{a+b}{2}\right)^2 - ab$:

$$= \frac{a^2+2ab+b^2}{4} - ab$$

$$= \frac{a^2+2ab+b^2-4ab}{4}$$

$$= \frac{a^2-2ab+b^2}{4}$$

$$= \frac{(a-b)^2}{4}$$

$$\geq 0$$

Hence $\left(\frac{a+b}{2}\right)^2 \geq ab$

Taking the positive square root of both sides gives $\frac{a+b}{2} \geq \sqrt{ab}$.

The triangle inequality

The triangle inequality states that $|x|+|y| \geq |x+y|$.

Proof

Remember that if $|x| \leq a$ then $-a \leq x \leq a$.

If x and y are either both positive or both negative, then $|x|+|y| = |x+y|$.

Now $-|x| \leq x \leq |x|$ since x equals either $|x|$ or $-|x|$.

Similarly, $-|y| \leq y \leq |y|$ since y equals either $|y|$ or $-|y|$.

Adding these two results gives: $-|x|-|y| \leq x+y \leq |x|+|y|$

$$-(|x|+|y|) \leq x+y \leq |x|+|y|$$

Let $a = x+y$ and $b = |x|+|y|$: $-b \leq a \leq b$

Hence $|a| \leq b$

So $|x+y| \leq |x|+|y|$

Or $|x|+|y| \geq |x+y|$ as required.